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# Viscosity Solutions of Cauchy Problems for Hamilton-Jacobi Equations

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The viscosity solutions of the Cauchy problem $u_t+H(x,u,Du)=0$, $u(x,0)=u_0(x)$ in $\boldsymbol{R}^N$, where $H: \boldsymbol{R}_N\times \boldsymbol{R}\times\boldsymbol{R}^N\to\boldsymbol{R}$ is a continuous function, are considered. We prove an existence and uniqueness theorem under a condition which is more general than the usual one with respect to the $u$ dependence of the Hamiltonian $H(x,u,p)$. This generalized condition would not necessarily guarantee that the stationary problem $u+H(x,u,Du)=f$ in $\boldsymbol{R}^N$ has a continuous viscosity solution. Our main method is based on the technique from nonlinear semigroup theory.

## 1. Introduction

In this paper we are concerned with the existence and uniqueness of viscosity solutions of the Cauchy problem for Hamilton-Jacobi equations

$$u_t+H(x,u,Du)=0 \quad \text{in } \boldsymbol{R}^N\times(0,T), \quad (1.1)$$

$$u(x,0)=u_0(x) \quad \text{in } \boldsymbol{R}^N, \quad (1.2)$$

in which $H: \boldsymbol{R}^N\times\boldsymbol{R}\times\boldsymbol{R}^N\to\boldsymbol{R}$ is continuous and $Du$ denotes the spatial gradient $(u_{x_1},\cdots,u_{x_N})$ of $u$. Crandall and Lions [3, 4], Crandall, Evans and Lions [1], Ishii [5, 6, 7] and Souganidis [10] have treated the problem (1.1)-(1.2) as well as the related stationary problem

$$u+\lambda H(x,u,Du)=f \quad \text{in } \boldsymbol{R}^N, \quad (1.3)$$

in which $\lambda>0$ and $f: \boldsymbol{R}^N\to\boldsymbol{R}$ is a given function. The conditions on the Hamiltonian $H$ assumed in these papers cited above guarantee that there exist both continuous viscosity solutions of (1.1)-(1.2) and (1.3).

The main purpose of this paper is to generalize the usual condition on the $u$ dependence of the Hamiltonian $H(x,u,p)$ and to prove that the Cauchy problem (1.1)-(1.2) has a unique and continuous viscosity solution under the condition (see the condition (H1) below); however, the uniqueness of viscosity solutions of the stationary problem (1.3) is not necessarily proved under the condition.

Another purpose of this paper is to give a nonlinear semigroup approach to the Cauchy problem (1.1)-(1.2) which is not a direct application of Crandall-Liggett's theorem, since the operator $A$ in $BUC(\boldsymbol{R}^N)$ defined formally by $Au=H(\cdot,u,Du)$ is "not necessarily accretive" in $BUC(\boldsymbol{R}^N)$ under the condition (H1). Our main tools are refinements of the technique from nonlinear semigroup theory (see [8], [9] and [11]).

## 2. Statements of Results

We will use the following notational conventions throughout. We will set $\Omega=\boldsymbol{R}^N$ and denote by $BUC(\Omega)$ the Banach space of bounded uniformly continuous functions defined on $\Omega$ with the maximum norm $\|u\|=\sup_{x\in\Omega}|u(x)|$. We denote by $\Gamma$ the set of functions $m:[0,\infty)\to[0,\infty)$ which are continuous, nondecreasing and subadditive and satisfy $m(0)=0$.

We will not explain the definitions of viscosity subsolutions, supersolutions and solutions of (1.1) and (1.3); we refer the reader to Crandall, Evans and Lions [1] and Ishii [7] for the precise definitions.

For the Hamiltonian $H\in C(\Omega\times\boldsymbol{R}\times\Omega)$ considered here we will assume the following conditions:

(H1) There is $\gamma\in\Gamma$ such that $\int_0^1 dr/\gamma(r)=\infty$ and

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$$H(x,r,p)-H(x,s,p)\geq -\gamma(r-s) \qquad (2.1)$$

for all $x,p\in\Omega$ *and* $r,s\in\boldsymbol{R}$ with $r\geq s$.

(H2) There is $\sigma\in\Gamma$ such that

$$|H(x,r,p)-H(x,r,q)|\leq\sigma(|p-q|) \qquad (2.2)$$

for all $x,p,q\in\Omega$ and $r\in\boldsymbol{R}$.

(H3) There is $m\in\Gamma$ such that

$$|H(x,r,p)-H(y,r,p)|\leq m(|x-y|(|p|+1))$$

for all $x,y,p\in\Omega$ and $r\in\boldsymbol{R}$.

(H4) $\sup_{x\in\Omega}|H(x,0,0)|<\infty$.

Our result is as follows:

Theorem. *Let* (H1)–(H4) *be satisfied and* $T>0$. *If* $u_0\in BUC(\Omega)$ *then there exists a unique viscosity solution* $u\in C([0,T];BUC(\Omega))$ *of* (1.1)–(1.2).

*Remark.* In [4] (also see [3] and [10]) the above theorem is proved under the conditions (H1)′, (H2)′, (H3) and (H4), where

(H1)′ There is a real number $a$ such that (2.1) holds with $\gamma(r)=ar$;

(H2)′ For each $R>0$ there is a $\sigma_R\in\Gamma$ such that (2.2) with $\sigma=\sigma_R$ holds for all $r\in\boldsymbol{R}$ and all $x,p,q\in\Omega$ with $|p|,|q|\leq R$.

Clearly, (H1)′ implies (H1) and (H2) implies (H2)′. Therefore, our result generalizes the condition on the $u$ dependence of the Hamiltonian $H$ at the expence of strengthening the condition of the Hamiltonian $H$ on the gradient of $u$. The condition (H2) is, however, only needed in the proof of Proposition 3.1 below; we may replace the condition (H2) by the usual condition (H2)′ in the other parts of Section 3 below. On the other hand, it would seem that the finiteness of the right derivative of $\gamma$ at 0 in the condition (H1)′ plays an important role in the arguments used in [4], [3] and [10]. We can, however, choose a function $\gamma\in\Gamma$ which satisfies $\int_0^1 dr/\gamma(r)=\infty$ and $\gamma'(+0)=\infty$ in the condition (H1), for example, $\gamma(r)=-r\log r$ for $0<r\leq(2e)^{-1}$ and $\gamma(r)=r\log 2+(2e)^{-1}$ for $r>(2e)^{-1}$.

## 3. Proof of Theorem

We start with the following comparison result for (1.3) which will play an important role in our arguments.

Proposition 3.1. *Let* (H1)–(H3) *be satisfied. Let* $\lambda>0$ *and* $u,v,f,g\in L^\infty(\Omega)$. *If* $u$ *and* $v$ *are a viscosity subsolution of* $u+\lambda H(x,u,Du)=f$ *on* $\Omega$ *and a viscosity supersolution of* $v+\lambda H(x,v,Dv)=g$ *on* $\Omega$, *respectively, then we have*

$$\sup_\Omega(u^*-v_*)^+\leq\lambda\gamma(\sup_\Omega(u^*-v_*)^+)+\sup_\Omega(f^*-g_*)^+.$$

*Here* $r^+$ *denotes* $\max\{r,0\}$ *and* $u^*$ *is the upper semicontinuous relaxation of* $u$ *defined by*

$$u^*(x)=\lim_{\varepsilon\to+0}\sup\{u(y);|y-x|<\varepsilon\} \quad \textit{for } x\in\Omega$$

*and* $u_*$ *is the lower semicontinuous relaxation of* $u$ *defined by* $u_*=-(-u)^*$.

Proof. This proposition is proved in [5] under the conditions that $u,v,f,g\in UC(\Omega)$ and (H1)′, (H2)′ and (H3) are satisfied. We can not, however, find the literature which gives the proof in the case that $u,v,f,g$ are discontinuous and (H1)′ is replaced by (H1). Hence we will give a brief proof. We follow the idea mentioned in [2]. Set $z(x,y)=u^*(x)-v_*(y)$ and $h(x,y)=f^*(x)-g_*(y)$ for $x,y\in\Omega$. Let $g_R\in C^1(\boldsymbol{R})$ satisfy $0\leq g_R'\leq 1$, $g_R(r)/r\to 1$ as $r\to\infty$ and $g_R(r)=0$ for $r\leq R$. Let $\beta,\eta,\varepsilon,\delta\in(0,1]$ and set

$$G=\{(x,y);x,y\in\Omega,|x-y|<\eta,|x|<r_\beta\}$$

with $r_\beta>R$ satisfying $a<\beta g_R(r_\beta)$, where $a=\sup_{x,y\in\Omega}z(x,y)$. We introduce the function which will correspond to $w_\varepsilon$ in the condition (H4) in [2]

$$w(x,y)=A\langle x-y\rangle_\delta^\mu+B+\beta g_R(|x|)+\lambda\gamma(\sup_G z^+)+\sup_G h^+$$

with $\langle x\rangle_\delta=(|x|^2+\delta^2)^{1/2}$, $\mu=\min\{\varepsilon/2\lambda m(\varepsilon),1\}$, $A=\max\{a/\eta^\mu, 2\lambda m(\varepsilon)/\varepsilon\}$ and $B=\lambda m(\varepsilon)+\lambda\sigma(\beta)+\varepsilon$. Comparing $w$ with $z$ on $G$ in a manner similar to [2], we will conclude

$$z\leq w \quad \text{on } G. \qquad (3.1)$$

Next we claim that for each $\beta>0$

$$\varlimsup_{\eta\to+0}\sup_G z\leq\sup_{x\in\Omega}z(x,x)^+ \qquad (3.2)$$

Indeed, $z^+$ attains its maximum at $(x_\eta,y_\eta)\in\bar{G}$. Since $v_*$ is l.s.c. and bounded from below, by Baire's theorem there exists a sequence $\{\varphi_n\}$ in $C(\Omega)$ such that $\varphi_1(y)\leq\varphi_2(y)\leq\cdots\leq\varphi_n(y)\nearrow v_*(y)$

for $y\in\Omega$. Since we may assume that $x_\eta\to x_0$ and $|\varphi_n(x_\eta)-\varphi_n(y)|\to 0$ as $\eta\to+0$, one has

$$\varlimsup_{\eta\to+0}\sup_G z^+\le\varlimsup_{\eta\to+0}(u^*(x_\eta)-\varphi_n(y_\eta))^+$$
$$\le(u^*(x_0)-\varphi_n(x_0))^+ .$$

By letting $n\to\infty$ we have (3.2). It is clear that (3.2) with $z$ replaced by $h$ is also valid.

We let $x=y$ in (3.1) and then $\delta\to+0$, $\varepsilon\to+0$, $\eta\to+0$, $\beta\to+0$ and $R\to\infty$ in this order. Then, using (3.2) we have the desired inequality.

Q.E.D.

Let $\lambda\in(0,1/2\gamma(1))$, $T>0$ and $N_\lambda=\{1,2,\cdots,[T/\lambda]+1\}$, where $[T/\lambda]$ denotes the largest integer part of $T/\lambda$. Let $u_0\in L^\infty(\Omega)$ and $u_k^\lambda\in L^\infty(\Omega)$, $k\in N_\lambda\cup\{0\}$, be the functions such that $u_0^\lambda=u_0$ and $u_k^\lambda$ is a viscosity solution of

$$u_k^\lambda+\lambda H(x,u_k^\lambda,Du_k^\lambda)=u_{k-1}^\lambda \quad\text{on } \Omega. \qquad (3.3)$$

Indeed, by (H1) and (H4) the equation (3.3) possesses viscosity sub- and supersolutions which take constant values on $\Omega$ whenever $u_{k-1}^\lambda\in L^\infty(\Omega)$. Thus, there exists a viscosity solution $u_k^\lambda$ of (3.3) in $L^\infty(\Omega)$ by virtue of [7, Theorem 3.1].

Now consider the Cauchy problem of ODE

$$\varphi'(t)=\gamma(\varphi(t)),\ t\ge 0,\quad \varphi(0)=\alpha>0.$$

Since $\gamma(r)$ is continuous and grows at most linearly as $r\to\infty$, Peano's theorem assures the existence of the global maximal solution $\bar\varphi(t;\alpha)$, $t\in[0,\infty)$. We set

$$\omega(\alpha)=\bar\varphi(T;\alpha). \qquad (3.4)$$

Then we have that $\bar\varphi(t;\alpha)\le\omega(\alpha)$ for $t\in[0,T]$ and $\lim_{\alpha\to+0}\omega(\alpha)=0$ since $\int_0^1 dr/\gamma(r)=\infty$.

Lemma 3.2. *Let $u_0\in BUC(\Omega)$. There exists a constant $M>0$ such that $OS(u_k^\lambda)\le\omega(M\lambda)$ for $k\in N_\lambda$, where $OS(u)=\sup_{x\in\Omega}(u^*(x)-u_*(x))$.*

Proof. By Proposition 3.1 we have

$$OS(u_k^\lambda)\le\lambda\gamma(OS(u_k^\lambda))+OS(u_{k-1}^\lambda) \qquad (3.5)$$

which yields that $OS(u_k^\lambda)\le(1-\lambda\gamma(1))^{-1}\{\lambda\gamma(1)+OS(u_{k-1}^\lambda)\}\le e^{2\gamma(1)T}T\gamma(1)$ since $OS(u_0)=0$. Define a step function $\phi_1$ on $[0,T]$ by $\phi_1(0)=0$ and $\phi_1(t)=OS(u_k^\lambda)$ for $t\in((k-1)\lambda,k\lambda]$, $k\in N_\lambda$. Then, summing (3.5) yields that

$$\phi_1(t)\le\int_0^t\gamma(\phi_1(\tau))d\tau+\lambda M,\quad t\in[0,T],$$

where $M=\gamma(e^{2\gamma(1)T}T\gamma(1))$. Therefore, by the maximum principle one has that $\phi_1(t)\le\bar\phi(t;\lambda M)\le\omega(M\lambda)$, $t\in[0,T]$. Q.E.D.

For $\lambda,\mu>0$ we set

$$a_{k,j}=\max\{\sup_\Omega(u_k^{\lambda *}-u_{j*}^\mu)^+,\sup_\Omega(u_j^{\mu *}-u_{k*}^\lambda)^+\}$$

and define step functions $u^\lambda$ and $u^{\mu,\lambda}$ on $[0,T]$ by $u^\lambda(0)=u^{\mu,\lambda}(0)=0$ and $u^\lambda(t)=u_k^\lambda$ and $u^{\mu,\lambda}(t)=u^\mu(k\lambda)$ for $t\in((k-1)\lambda,k\lambda]$, $k\in N_\lambda$.

Lemma 3.3. *Let $u_0\in BUC(\Omega)$ and $w\in C_0^\infty(\Omega)$. Then there exist constants $C$ and $C_w$ and $\rho,\rho_w\in\Gamma$ such that $C$ and $\rho$ are independent of $w$ but $C_w$ and $\rho_w$ may depend on $w$ and that*

$$\begin{aligned}a_{k,j}&\le C\|u_0-w\|+C_wf_{k,j}\\&+\int_0^{k\lambda}\gamma(\|u^\lambda(\tau)-u^{\mu,\lambda}(\tau)\|)d\tau\\&+j\mu\{\rho(\|u_0-w\|)+\rho_w(\lambda+\mu)\\&+\delta^{-1}\rho_w(T)f_{k,j}+\rho_w(2\delta)\}\end{aligned}\qquad (3.6)$$

*for all $\delta\in(0,T/2)$, $\lambda,\mu\in(0,\delta)$, $k\in N_\lambda$, $j\in N_\mu$. Here*

$$f_{k,j}=\{(k\lambda-j\mu)^2+k\lambda^2+j\mu^2\}^{1/2}.$$

Proof. In what follows $C$, $C_w$ and $\rho$, $\rho_w$ *will denote various constants and functions in $\Gamma$, respectively, such that* $C$ and $\rho$ are independent of $w$ but $C_w$ and $\rho_w$ may depend on $w$. It follows from Proposition 3.1 that $\sup_\Omega(u_k^{\lambda *}-w)^+\le(1-\lambda\gamma(1))^{-1}\{\lambda\gamma(1)+\sup_\Omega(u_{k-1}^{\lambda *}-f)^+\}$ where $f=w+\lambda H(x,w,Dw)$. Hence we have that

$$\sup_\Omega(u_k^{\lambda *}-w)^+\le C\|u_0-w\|+C_wk\lambda. \qquad (3.7)$$

Similarly, $\sup_\Omega(w-u_{k*}^\lambda)^+$ has the same bound. Thus one has that $\|u_k^\lambda\|\le C_w$ for $k\in N_\lambda$. By Proposition 3.1, for each $i\in N_\lambda$

$$d_k=\max\{\sup_\Omega(u_{k+i}^{\lambda *}-u_{k*}^\lambda)^+,\sup_\Omega(u_k^{\lambda *}-(u_{k+i}^\lambda)_*)^+\}$$

satisfies

$$d_k\le d_{k-1}+\lambda\gamma(d_k)\le d_0+\lambda\sum_{j=1}^k\gamma(d_j),$$

which together with (3.7) and Lemma 3.2 implies that

$$\|u^\lambda_{k+i}-u^\lambda_k\|\leq \Lambda_1+\sum_{j=1}^{k-1}\gamma(\|u^\lambda_{j+i}-u^\lambda_j\|),$$

where $\Lambda_1=C\|u_0-w\|+C_w i\lambda+\rho_w(\lambda)$. Therefore, by the maximum principle one has

$$\|u^\lambda_{k+i}-u^\lambda_k\|\leq w(\Lambda_1) \quad \text{for } k,k+i\in N_\lambda. \quad (3.8)$$

Next, from Proposition 3.1 one has that

$$(\lambda+\mu)a_{k,j}\leq\lambda a_{k,j-1}+\mu a_{k-1,j}+\lambda\mu\gamma(a_{k,j}).$$

On the other hand, by (3.8) and Lemma 3.2 one has that

$$\gamma(a_{k,j})\leq\gamma(\|u^\lambda(k\lambda)-u^{\mu,\lambda}(k\lambda)\|)+\Lambda_2.$$

where $\Lambda_2=\rho(\|u_0-w\|)+\rho_w(\lambda+\mu)+\rho_w(|k\lambda-j\mu|)$. Thus we obtain

$$(\lambda+\mu)a_{k,j}\leq\lambda a_{k,j-1}+\mu a_{k-1,j} +\lambda\mu\{\gamma(\|u^\lambda(k\lambda)-u^{\mu,\lambda}(k\lambda)\|)+\Lambda_2\}. \quad (3.9)$$

To prove (3.6) we first note that (3.6) is valid for $k=0$ or $j=0$ by (3.7). Therefore, by using (3.9) and the usual induction arguments as in [8, 9] we conclude that (3.6) is valid for all $k\in N_\lambda$ and $j\in N_\mu$. *Q.E.D.*

PROOF OF THEOREM. We first fix $\lambda,\mu>0$ arbitrarily. We then take $\delta=(\lambda+\mu)^{1/4}$ and integers $k,j$ such that $j\mu\in((k-1)\lambda,k\lambda]$ in (3.6). Then, by using (3.8) one easily obtains

$$\|u^\lambda(t)-u^{\mu,\lambda}(t)\|\leq\rho(\|u_0-w\|)+\rho_w(\lambda+\mu)+\int_0^t\gamma(\|u^\lambda(\tau)-u^{\mu,\lambda}(\tau)\|)d\tau$$

with suitable $\rho,\rho_w\in\Gamma$. Note that $C_0^\infty(\Omega)$ is dense in $BUC(\Omega)$, and $w$ can be taken arbitrarily from $C_0^\infty(\Omega)$. Hence it follows from the fact that $\int_0^1 dr/\gamma(r)=\infty$ that

$$\lim_{\lambda,\mu\to+0}\sup_{t\in[0,T]}\|u^\lambda(t)-u^{\mu,\lambda}(t)\|=0,$$

which together with (3.8) implies that there exists a function $u\in C([0,T];L^\infty(\Omega))$ (see [9, 10]) such that

$$\lim_{\lambda\to+0}\sup_{t\in[0,T]}\|u(\cdot,t)-u^\lambda(t)\|=0.$$

Since $OS(u(\cdot,t))=0$ for $t\in[0,T]$ by Lemma 3.2, $u$ must lie in $BC(\Omega\times[0,T])$, which denotes the set of bounded continuous functions defined on $\Omega\times[0,T]$. Then, the proof of [1, Proposition 5.2] is easily adapted to prove that $u$ is a viscosity solution of (1.1)–(1.2). Therefore the proof of the theorem will be completed if we prove the following lemma. *Q.E.D.*

LEMMA 3.4. *Let* (H1)–(H3) *be satisfied. If* $u_0\in BC(\Omega)$ *then* (1.1)–(1.2) *has at most one viscosity solution* $u$ *in* $BC(\Omega\times[0,T])$. *Moreover, if* $u_0\in BUC(\Omega)$ *then* $u\in C([0,T];BUC(\Omega))$.

OUTLINE OF PROOF. Let $u$ and $v$ be two viscosity solutions of (1.1) in $BC(\Omega\times[0,T])$ and $z(x,y,t)=u(x,t)-v(y,t)$. Let $g_R, G$ and so on be the functions as in the proof of Proposition 3.1. Consider the function

$$z(x,y,t)=\{A\langle x-y\rangle_\delta^\mu+B+\beta g_R(|x|)\}(t+1)+\int_0^t\gamma(\sup_G z(\cdot,\cdot,s)^+)ds+\sup_G z(\cdot,\cdot,0)^+,$$

where $\mu=\min\{\varepsilon/2(T+1)m(\varepsilon),1\}$, $A=\max\{a/\eta^\mu, 2m(\varepsilon)/\varepsilon\}$ and $B=m(\varepsilon)+\sigma(\beta(T+1))+\varepsilon$. By the comparison argument as in [2] we obtain

$$z\leq w \quad \text{on } G\times(0,T]. \quad (3.10)$$

In (3.10) we let $x=y$ and then $\delta\to+0$, $\varepsilon\to+0$, $\eta\to+0$, $\beta\to+0$ and $R\to\infty$ in this order. Then we have that on $\Omega\times[0,T]$

$$z(x,y,t)^+\leq\sup_{x\in\Omega}z(x,x,0)^+ +\int_0^t\gamma(\sup_{x\in\Omega}z(x,x,s)^+)ds.$$

Here we have used (3.2) with $z(x,y)$ replaced by $z(x,y,t)$. This implies that if $z(x,x,0)^+=0$ then $z(x,x,t)^+=0$ for $t\in[0,T]$. Hence we have the uniqueness of solutions.

Finally, we let $u=v$ in (3.10) and then let $r_\beta\to\infty$, $\eta\to+0$, $\delta\to+0$, $\varepsilon\to+0$, $\beta\to+0$ and $R\to\infty$ in this order. Then one obtains that the function $\xi(t)=\lim_{\eta\to+0}\sup_{|x-y|<\eta}|u(x,t)-v(y,t)|$ satisfies

$$\xi(t)\leq\xi(0)+\int_0^t\gamma(\xi(s))ds.$$

Since $\xi(0)=0$ by $u_0\in BUC(\Omega)$, we have that $\xi(t)=0$ and hence $u(\cdot,t)\in BUC(\Omega)$ for $t\in[0,T]$. *Q.E.D.*

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