MEMOIRS OF SHONAN
INSTITUTE OF TECHNOLOGY
Vol. 27, No. 1, 1993

Deviation of Geodesics in the Gravitational Field of Finslerian Space-Time

P. C. STAVRINOS* and H. KAWAGUCHI**

一般相対論によれば，重力の影響下にある質点の運動は 4 次元リーマン空間の測地線の微分方程式によって記述される。このとき，リーマン計量は重力ポテンシャルを表わしている。さらに，重力場のなかで質点の代わりに有限のひろがりをもった物体を自由落下させるとき潮汐力が生ずる。これは，測地的偏差によって表現され，その加速度はリーマン・クリストッフェルの曲率テンソルで記述される。

この論文では，上記のリーマン空間を n 次元のフィンスラー空間へ一般化した場合の測地的偏差を与える微分方程式について研究し，その成果を拡張してフィンスラー空間の接触リーマン空間における微分方程式を求める。さらに，一般化されたフィンスラー空間の場合にも測地的偏差の微分方程式を検証する。最後に，定曲率の接触リーマン空間における測地的偏差の微分方程式を導く。

1. Introduction

The profound role of the equation of Riemannian geodesic deviation has been recognized by the general relativity for a long time (cf. Ref. 11)). It is known that if there are deviations in geodesically moving free particles, they will be caused by the curvature of the space, which physically is interpreted by the existence of tidal forces. The relative accelerations of nearby time-like geodesics are caused by the curvature of the space-time. In Riemannian spaces, the curvature tensor $R_j{}^i{}_{kl}$ enters fully in the equation of geodesic deviation and it produces the relative accelerations.

In the Finslerian approach, the curvature of a Finsler space-time is characterized not only by the tensor $R_j{}^i{}_{kl}$ but also by the tensors $S_j{}^i{}_{kl}$, $P_j{}^i{}_{kl}$ and $K_j{}^i{}_{kl}$ (Ref. 2), 6), 8)). Thus, the question arises when it is possible to find a full interpretation of the curvature of a Finsler space in terms of geodesic deviations. These considerations will be presented in another work.

In the present paper, we study the deviation equation in Finsler spaces. In paragraph §2 we give an interpretation of the equation of geodesic deviation as it has given by H. Rund (Ref. 8). Also, in §3 we extend the form of this equation for the tangent Riemannian space of a Finsler space, and we examine the deviation of geodesics for a generalized Finsler metric $g_{ij}=\alpha_{ij}+\beta h_{ij}$ (see Ref. 1)). Finally in §4, considering a tangent Riemannian space of constant curvature, we derive the equation of geodesic deviations.

2. Tidal Forces in a Finsler Space-Time

We consider a Finsler space-time, with fundamental function F and metric tensor g_{ij} which is given by,

$$g_{ij}=\frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}, \qquad y^i=\frac{\mathrm{d}x^i}{\mathrm{d}t} \tag{2.1}$$

and $\mathrm{d}s^2=g_{ij}(x,y)\mathrm{d}x^i\mathrm{d}x^j$, then the movement of a free particle on a geodesic in the gravitational field, is derived by the variational principle

$$\delta\int \mathrm{d}s=0 \tag{2.2}$$

which implies the equation of the geodesics in the Finsler space-time,

* Associate Professor, Department of Mathematics, University of Athens, 15771, Greece. (1992 年 5 月 11 日　本学情報工学科を訪問し講演した。)
** 情報工学科　助教授
平成 4 年 9 月 28 日受付

$$x''^i+\gamma_h{}^i{}_k(x, y)x'^h x'^k=0 \qquad (2.3)$$

with

$$x'^j=\frac{dx^j}{ds}, \qquad x''^j=\frac{dx'^j}{ds}$$

and the Christoffel symbols,

$$\gamma_{ihk}=\frac{1}{2}\left[\frac{\partial g_{hk}}{\partial x^i}+\frac{\partial g_{ih}}{\partial x^k}-\frac{\partial g_{ki}}{\partial x^h}\right]. \qquad (2.4)$$

From a more general viewpoint, a Finsler space is a fibered space, so that the movement of a particle in a Finsler tangent bundle is represented by the curve $(x^i(t), y^i(t))$, which is distinguished in the horizontal h-path on the base manifold and the vertical v-path on the tangent space. Therefore, introducing the generalized element of length

$$d\sigma^2=g_{ij}(x, y)dx^i dx^j+g_{ij}(x, y)Dy^i Dy^j \qquad (2.5)$$

where

$$Dy^i=dy^i+N^i_j(x, y)dx^j,$$

the geodesics will be derived by the variational principle,

$$\left.\begin{aligned}&\delta\int d\sigma=0\\ &\frac{d^2x^i}{d\sigma^2}+F_j{}^i{}_k(x, x')\frac{dx^j}{d\sigma}\frac{dx^k}{d\sigma}\\ &\qquad+C_j{}^i{}_k(x, x')\frac{dx^j}{d\sigma}\frac{d^2x^k}{d\sigma^2}=0.\end{aligned}\right\} \qquad (2.6)$$

Deviations of such curves $(x^i(t), y^i(t))$ will be examined in a separate paper. The geodesic deviation in the framework of the fibered Finslerian gauge approach has been studied in Ref. 3).

In the following, we shall get the equation of geodesic deviation of a Finsler space, with respect to the third curvature tensor $K_j{}^i{}_{hk}$ of the space. An interpretation of the geodesic deviation of two neighbouring free particles, moving in the Finslerian gravitational field, can be given by their relative accelerations. So it is possible to be revealed the tidal forces with relation to the curvature $K_j{}^i{}_{hk}$. In this case we follow H. Rund (see Ref. 8)).

Let $x^i(\lambda, s)\subset F^4$ be a two-parameter family of geodesics in a Finsler space-time where λ is the affine parameter or the proper time, and the s denotes the family of geodesics, then we assume the equation of geodesic deviation,

$$\frac{\delta^2 z^i}{\delta\lambda^2}+K_j{}^i{}_{hk}(x, \xi)\xi^j z^h\xi^k=0 \qquad (2.7)$$

where z means the deviations vector, which measures the relative acceleration between two neighbouring particles.

$$\xi^i=\frac{\partial x^i}{\partial\lambda}, \quad \frac{\delta z^i}{\delta\lambda}=z^i{}_{|h}\xi^h=\left[\frac{\partial z^i}{\partial x^h}+F_h{}^i{}_k(x, y)z^k\right]\xi^h$$

$$\begin{aligned}K_j{}^i{}_{hk}(x, y)=&\left[\frac{\partial F_j{}^i{}_h}{\partial x^k}-\frac{\partial F_j{}^i{}_h}{\partial y^l}\frac{\partial G^l}{\partial x^k}\right]\\ &-\left[\frac{\partial F_j{}^i{}_k}{\partial x^h}-\frac{\partial F_j{}^i{}_k}{\partial y^l}\frac{\partial G^l}{\partial y^h}\right]\\ &+F_m{}^i{}_k F_j{}^m{}_h-F_m{}^i{}_h F_j{}^m{}_k\end{aligned}$$

$$G^i(x, y)=\frac{1}{2}\gamma_h{}^i{}_k(x, y)y^h y^k$$

$$F_{kij}=\gamma_{kij}-C_{jih}\frac{\partial G^h}{\partial y^k}-C_{kih}\frac{\partial G^h}{\partial y^j}+C_{kjh}\frac{\partial G^h}{\partial y^i}.$$

Hence, if the vector $\frac{\delta^2 z^i}{\delta\lambda^2}$ is different from zero, then it will imply $K_j{}^i{}_{hk}\neq 0$. A physical explanation of $\frac{\delta^2 z^i}{\delta\lambda^2}$ can be given by the tidal forces in the gravitational field of the Finsler space. The condition $K_j{}^i{}_{hk}\neq 0$ is equivalent to the existence of

$$(R^{\mu\nu}-g^{\mu\nu}R)_{|\mu}=U^\nu\neq 0 \qquad (2.8)$$

$$U_\nu=\frac{1}{2}g^{i\lambda}(P_i{}^\kappa{}_{\lambda\alpha}R_\kappa{}^\alpha{}_\nu+P_i{}^\kappa{}_{\kappa\alpha}R_\nu{}^\alpha{}_\lambda+P_i{}^\kappa{}_{\nu\alpha}R_\lambda{}^\alpha{}_\kappa)$$

and we have considered $T^{\mu\nu}{}_{|\mu}=\frac{1}{\kappa}U^\nu$ (cf. Ref. 12)). The vector U^ν acts as a source or sink well or as force density of the energy-momentum tensor $T_{\mu\nu}$.

When the relative acceleration between the paths is zero, (tidal forces are zero) in all families of geodesics of the Finsler space-time we have

$$K_j{}^i{}_{kl}=0. \qquad (2.9)$$

If in a point P of the space, the relative accelerations of the nearby time-like geodesics become infinite, it means that there is a "physical" singularity in that point and also the curvature becomes infinite.

In spaces, where the dimension is larger than two, the deviation vector when it moves along the fundamental geodesic, remains vertical on this path, but it also turns round this one (Ref. 7), p. 29).

3. Deviation of Vertical Geodesics in a "Tangent Riemannian" Space-Time

We shall restrict our study in a tangent Riemannian space-time (x; constant), with metric,

$$ds_x{}^2 = g_{ij}(x, y)dy^i dy^j \ .$$

The gravitational field in a Finsler space-time $F_x{}^{(4)}$ is governed by the Riemannian curvature tensor $S_h{}^i{}_{jk}$ which is derived from the Cartan connection $C_j{}^i{}_k(y)$.

Let $F_x{}^{(2)}$ be a two-dimensional geodesic surface of the Finsler space-time $F_x{}^{(4)}$, and suppose that $F_x{}^{(2)}$ may be represented parametrically by the equations

$$y^i(u, U) \quad /i=1, 2, 3, 4$$

where u, U are the Gaussian parameters of the surface. The functions $y^i(u, U)$ being of class C^4, we denote the tangent vectors of the parameters lines, u and U, by n^i, ξ^i, respectively

$$\xi^i = \frac{\partial y^i(u, U)}{\partial u} , \quad n^i = \frac{\partial y^i(u, U)}{\partial U} \tag{3.1}$$

so that we get

$$\frac{\partial \xi^i}{\partial U} = \frac{\partial^2 y}{\partial u \partial U} = \frac{\partial n^i}{\partial u} \ . \tag{3.2}$$

Now, we shall derive the equation of the vertical geodesic deviation on $F_x{}^{(4)}$ (x: const). Two nearby geodesics C, C' on $F_x{}^{(2)}$ as above, will be satisfied by the equation,

$$\frac{d^2 y^i}{ds^2} + C_m{}^i{}_n(x, y)\frac{dy^m}{ds}\frac{dy^n}{ds} = 0$$

with $ds = F(x, dy)$, (the parameter $s=u$ plays the role of affine parameter or proper time) and,

$$\zeta^k = \frac{\partial y^k}{\partial u} du + \frac{\partial y^k}{\partial U} dU \tag{3.3}$$

is the deviation vector between C and C'. The parameter U will symbolize the deviation from a base geodesic to its infinitesimally nearby geodesic.

The covariant derivative and the commutation relation of a vector field $X^i(y^k)$ on a tangent Riemannian space, have the following form,

$$X^i|_h = \frac{\partial X^i}{\partial y^h} + C_h{}^i{}_k X^k \ , \tag{3.4}$$

$$X^i|_h|_k - X^i|_k|_h = S_j{}^i{}_{kh} X^j \ . \tag{3.5}$$

Using the notation D_j, it is known that

$$|_j = F D_j \tag{3.6}$$

so that we can write for the vector field ξ^i,

$$D_j \xi^i = \frac{1}{F}\left[\frac{\partial \xi^i}{\partial y^j} + C_j{}^i{}_k \xi^k\right] . \tag{3.7}$$

Taking account of the relations (3.1) and (3.2), we get,

$$\xi^i{}_{|u} n^u = \left[\frac{\partial^2 y^i}{\partial u \partial U} + C_h{}^i{}_k \xi^h n^k\right] = n^i{}_{|u} \xi^u \tag{3.8}$$

or by virture of (3.7) we have

$$(D_k \xi^i) n^k = (D_k n^i) \xi^k \ . \tag{3.9}$$

If we consider the commutation relation (Ref. 2), p. 30), we get,

$$D_l D_m \xi^i - D_m D_l \xi^i = F^{-2} S_l{}^i{}_{mk} \xi^k \tag{3.10}$$

$$(D_l D_m \xi^i - D_m D_l \xi^i) \xi^l n^m = F^{-2} S_m{}^i{}_{lk} \xi^k \xi^l n^m \ . \tag{3.11}$$

The geodesics C and C' can be denoted by $U, U+\varepsilon$, where $\varepsilon = dU$ is constant. Along these paths

$$D_u \xi^i = 0 \quad \text{and} \quad (\partial/\partial U)(D_u \xi^i)$$

are valid so that we have

$$D_U D_u \xi^i = 0 \ .$$

If we substitute the above relations to (3.11), by virtue of (3.3) and (3.9), we find

$$(D_l D_m \xi^i)\xi^l n^m = F^{-2} S_l{}^i{}_{mk} n^l \xi^k \xi^m \ . \qquad (3.12)$$

By the relation (3.9), the equation (3.12) become

$$\varepsilon^2 \frac{D^2 n^i}{ds^2} + S_m{}^i{}_{lk} \xi^k n^m \xi^l F^{-2} = 0 \ , \qquad (3.13)$$

where $u^* - u = f(u) = ds$ is of the order of magnitude of ε, and we have chosen the function $f(u)$ so that $f''(u) = 0$. Hence, the relation (3.13) will take the form,

$$\frac{D^2 \zeta^i}{ds^2} + S_j{}^i{}_{kl} \xi^j \zeta^k \xi^l F^{-2} = 0 \ . \qquad (3.14)$$

Because the vector ξ^i is in the direction of the tangential line element y^i, (3.14) can be written,

$$\frac{D^2 \zeta^i}{ds^2} + F^{-2} S_j{}^i{}_{kl} \frac{dy^j}{ds} \zeta^k \frac{dy^l}{ds} = 0 \ . \qquad (3.15)$$

If we put $S_j{}^i{}_{kl} \xi^j \xi^l = S_0{}^i{}_{k0}$ in (3.14), we get

$$\frac{D^2 \zeta^i}{ds^2} + S_0{}^i{}_{k0} \zeta^k = 0 \ , \qquad (3.16)$$

$$\frac{D\theta^i}{ds} + S_0{}^i{}_{k0} \zeta^k = 0 \ , \qquad (3.17)$$

where $\theta^i = D\zeta^i/ds$: the components of the rotation of ζ^i, (cf. § 2). The relation (3.17) is analogous with the equation (47) of E. Cartan (Ref. 5)) with respect to the curvature tensor $S_j{}^i{}_{kl}$. The relation (3.14) is the equation of the vertical geodesic deviation in which the first term shows the relative acceleration between two test particles and the second one represents the tide-producing gravitational forces which are expressed in terms of Riemannian curvature tensor $S_j{}^i{}_{kl}$ of a tangent Riemannian space-time. It will be noted that the equation (3.14) can be related to the so-called Einstein's equations in the tangent space, i.e. relation (3.18).

The nullification of $\frac{D^2 \zeta^i}{ds^2}$ entails from (3.14) that $S_i{}^j{}_{mn} = 0$ and this occurs only when Ricci tensor $S_{ij} = 0$ (cf. Ref. 6), 9)). In this case, we have an "empty" region and the space in a Riemannian space. In the equation of the gravitational field of a tangent Riemannian space, as was proposed by Y. Takano (cf. Ref. 12)).

$$S_{ij} - \frac{1}{2} S g_{ij}(x, y) = \kappa T_{ij}(x, y) \ , \qquad (3.18)$$

where κ=internal gravitational constant, when $T_{ij}=0$, (T_{ij}=internal energy-momentum tensor) we have $S_{ij}=0$, which is equivalent to $S_j{}^i{}_{kl}=0$ (cf. Ref. 12)). Hence, the velocity $\frac{D\zeta^i}{ds}$ between the nearby particles is constant, the v-paths are zero and the follow particles h-paths are parallel with each other, on the base Riemannian manifold.

The tidal forces are dependent on the geometry of the space. Namely they are produced by the nature of the space itself. As a special case, we can restrict our study in the geodesic deviation between the geodesically moving particles, to the indicatrix I_3, namely for a Riemannian hypersurface of $F_x{}^{(4)}$, with equation $F(x, y) = 1$, x=const.

We consider the angle $d\varphi$, between the unitary tangent vectors of $F_x{}^{(4)}$, $d\varphi = (\widehat{y^k, y^k + dy^k})$, then $d\varphi$ is defined by,

$$d\varphi^2 = e \varphi_{\mu\nu} \, dy^\mu \, dy^\nu \qquad e = \pm 1 \qquad (3.19)$$

where $\varphi_{\mu\nu}$ is the angular metric,

$$\varphi_{\mu\nu} = \frac{1}{F} \frac{\partial^2 F}{\partial y^\mu \partial y^\nu} \qquad (3.20)$$

so that a vertical path on the indicatrix I_3 with φ as a parameter, will be governed by the equation

$$\frac{d^2 y^i}{d\varphi^2} + C_k{}^i{}_l(y) \frac{dy^k}{d\varphi} \frac{dy^l}{d\varphi} = 0 \ . \qquad (3.21)$$

On the indicatrix $d\varphi = ds$ (Ref. 8), p. 209). If $S_{\alpha\beta\gamma\delta}$ is the curvature tensor of I_3 then the equation of geodesic deviation can be written

$$\frac{D^2 z^i}{d\varphi^2} + F^{-2} S_\beta{}^\alpha{}_{\gamma\delta} b^{\beta\gamma\delta i}_{jkl\alpha} \frac{dy^j}{d\varphi} z^k \frac{dy^l}{d\varphi} = 0 \qquad (3.22)$$

$$b^{\beta\gamma\delta i}_{jkl\alpha} = Y^\beta_j Y^\gamma_k Y^\delta_l Y^i_\alpha \qquad \left[Y^i_\alpha = \frac{\partial y^i}{\partial u^\alpha} \ , \quad \begin{matrix} i = 1,2,3,4 \\ \alpha = 1,2,3 \end{matrix} \right]$$

where z^k presents the "angular deviation" vector and

$$S_{hijk} = S_{\alpha\beta\gamma\delta} Y^\alpha_h Y^\beta_i Y^\gamma_j Y^\delta_k \ .$$

In the above equation we can substitute the curvature tensor $S_{\beta}{}^{\alpha}{}_{\gamma\delta}$ with $R_{\beta}{}^{\alpha}{}_{\gamma\delta}$, where $R_{\beta}{}^{\alpha}{}_{\gamma\delta}$ is the Riemannian curvature on the indicatrix (cf. Ref. 6)),

$$S_{\alpha\beta\gamma\delta}=R_{\alpha\beta\gamma\delta}-(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma}) \ . \qquad (3.23)$$

Hence, the equation (3.22) becomes by virture of (3.23)

$$\frac{D^2z^i}{d\varphi^2}+F^{-2}(R_{\beta}{}^{\alpha}{}_{\gamma\delta}-g^{\alpha}_{\gamma}g_{\beta\delta}-g^{\alpha}_{\delta}g_{\beta\gamma}) \times b^{\beta\,\gamma\,\delta\,i}_{jkl\,\alpha}\frac{dy^j}{d\varphi}z^k\frac{dy^l}{d\varphi}=0 \ . \qquad (3.24)$$

Furthermore, we consider an interesting example of generalized metric $g_{ij}=\alpha_{ij}(x)+\beta h_{ij}(x,y)$ (cf. Ref. 1)), with $h_{ij}=\alpha_{00}^{-1}\alpha_{i0}\alpha_{j0}-\alpha_{ij}$, $\alpha_{i0}=\alpha_{ij}y^j$, $\alpha_{00}=\alpha_{i0}y^i$, β is a parameter and $F^2=g_{00}=\alpha_{00}=g_{ij}y^iy^j$.

We shall examine the form of equation (3.14) in the case of generalized tangent space $M(x)$. The Cartan-like symbols have been calculated (cf. Ref. 1), 10))

$$C_j{}^i{}_k=\frac{\beta\alpha^{ri}(\alpha_{00}^{-1}\alpha_{r0}\alpha_{jk}-\alpha_{00}^{-2}\alpha_{j0}\alpha_{r0}\alpha_{k0})+\beta^2(\alpha_{00}^{-1}h_{jk}y^i)}{1-\beta} \ . \qquad (3.25)$$

If we substitute (3.25) to

$$F^{-2}S_i{}^j{}_{mn}=C_k{}^j{}_mC_i{}^k{}_n-C_k{}^j{}_nC_i{}^k{}_m \qquad (3.26)$$

and after some calculations, we get the deviation equation,

$$\begin{aligned}(1-\beta)F^{-2}S_i{}^j{}_{mn}&=(1-2\alpha_{00}^{-2})X_m{}^j{}_{in}\\&+(\alpha_{00}^{-3}-\alpha_{00}^{-1})T_m{}^j{}_i\alpha_{n0}-\beta(\alpha_{00}E_m{}^j{}_{in}-\alpha_{00}^{-1}B_i{}^j{}_{mn})\\&-\beta^2\alpha_{00}^{-2}(\alpha_{k0}\alpha_{i0}\delta^j_n\delta^k_m+\beta^2A_i{}^j{}_{mn})\\&+\beta^3\alpha_{00}^{-2}H_i{}^j{}_{mn}-\alpha_{00}^{-2}T^j_{m\ in} \ ,\end{aligned}$$

where we have put,

$$\begin{aligned}&A_i{}^j{}_{mn}=H_k{}^j{}_mH_i{}^k{}_n-H_n{}^j{}_kH_m{}^k{}_i \ , \qquad H_i{}^k{}_j=h_{ij}y^k\\&H_i{}^j{}_{mn}=\alpha_{in}H_0{}^j{}_m+\delta^i_0\alpha_{km}H_i{}^k{}_n-\alpha_{i0}h^j_ny_m-\delta^j_nH_m{}^0{}_i\\&B_i{}^j{}_{mn}=T_i{}^k{}_mH_n{}^j{}_k+T_k{}^j{}_nH_m{}^k{}_i\\&T_j{}^i{}_k=X^i_j\alpha_{k0} \ , \qquad T_n{}^j{}_{im}=T_k{}^j{}_nT_i{}^k{}_m\\&E_m{}^j{}_{in}=T_m{}^j{}_kH_i{}^k{}_n\\&X^j_m=\beta^2\alpha_{00}^{-2}\delta^j_0\alpha_{m0} \ , \qquad X_m{}^j{}_{in}=X^j_m\alpha_{in} \ .\end{aligned}$$

4. Deviation of Vertical Geodesics in a Tangent Riemannian Space of Constant Curvature

Finsler spaces with constant curvature are important for geometry as well for Physics (cf. Ref. 4), 6)). In our case, we assume the deviation equations for a tangent Riemannian space M^n_x $(n\geq4)$ of constant curvature. The following is valid: (cf. Ref. 6) th. 31. 6 p. 225, Ref. 9))

$$F^2S_{nijk}=S(h_{nj}h_{ik}-h_{nk}h_{ij}) \qquad (4.1)$$

or

$$F^2S_i{}^l{}_{jk}=S(h_{ik}\delta^l_j-h_{ij}\delta^l_k) \ , \qquad S=\text{const.}$$

If we substitute this equation to (3.14) we get

$$\frac{D^2n^l}{ds^2}+S(h_{ik}\delta^l_j-h_{ij}\delta^l_k)v^iv^kn^j=0 \qquad (4.2)$$

where we symbolize the tangent vectors with v^i, v^k and n^j is the deviation vector. From (4.2) we have,

$$\frac{D^2n^l}{ds^2}+Sv_kv^kn^l-Sv_jn^jv^l=0 \ . \qquad (4.3)$$

If we apply the relation (4.3) for the gravitational field to the Finsler M^4_x with respect to nearby vertical geodesics of M^4_x as we have mentioned above, and considering the relations $v_iv^i=1$ and $v_kn^k=0$ then the equation (4.3) is reduced to the form,

$$\frac{d^2n^i}{ds^2}+Sn^i=0 \qquad (4.4)$$

which has the obvious solution

$$z^i=\sin A\,(\kappa\sqrt{S}) \qquad (4.5)$$

where the vector z^i is parallel to n^i.

In consequence of the equation (4.1) and Ref. 6) th. 31.6 p. 225, the indicatrix I_x of $M^{(4)}_x$ is a Riemannian space of constant curvature,

$$R_{\alpha\beta\gamma\delta}=(S+1)(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma}) \qquad (4.6)$$

where $R_{\alpha\beta\gamma\delta}$ is the curvature of the indicatrix and

$$g_{\alpha\beta}=h_{ij}Y^i_\alpha Y^j_\beta \quad \left[Y^i_\alpha=\frac{\partial y^i}{\partial u^\alpha}\right] .$$

Hence, repeating the above procedure, we take a similar result for the deviation equation of the indicatrix I_x,

$$\frac{d^2\theta^i}{ds^2}+(S+1)\theta^i=0 \qquad (4.7)$$

where we put θ^i the deviation vector between two infinitesimally nearby geodesics on the indicatrix I_x.

5. Conclusion

So in Riemannian space-time, as in the Finslerian approach of space-time, the existence of the gravitational field is manifested by the deviation of two-test particles, which are located to nearby geodesics. The study of geodesic deviation is closely related to the existence of tidal forces which are intrinsically produced by the geometry (curvature tensor) of the space. The kind of equation of geodesic deviation reveals physically the nature of these forces.

In the tangent Riemannian space the deviations of vertical geodesics are important because they imply the existence of the vertical component of the energy-momentum tensor $T^{int}_{\mu\nu}$ of the Einstein's equations, so that in case of "the empty vertical space", the vertical geodesics are free of any deviations.

References

1) A. K. Aringazing and G. S. Asanov: Rep. Math. Phys. **25** (1988), 183.
2) G. S. Asanov: Finsler Geometry Relativity and Gauge Theories, D. Reidel, Dordrecht 1985.
3) G. S. Asanov and P. C. Stavrinos: Rep. Math. Phys.
4) G. A. Asanov: Fortschr. Phys. **39** (1991) **3**, 185–210.
5) E. Cartan: Les espaces de Finsler, Actual. 79, Paris 1934.
6) M. Matsumoto: Foundations of Finsler Geometry and Special Finsler Spaces, Kaiseisha Pres. Kaiseisha 1986.
7) C. W. Misner-K. S. Thorne-I. A. Wheeler: Gravitation Publ. Freeman. Freeman, San Francisco 1973.
8) H. Rund: The Differential Geometry of Finsler Spaces, Springer, Berlin 1959.
9) H. Shimada: J. Korean Math. Soc. Vol. 14, No. 1, 1977.
10) H. Shimada: Symp. on Finsler Geometry at Awara Sept. 29, 1989.
11) J. L. Synge: Relativity: The general Theory, North Holland. Amsterdam 1960.
12) Y. Takano: Proc. Int. Symp. Rel. Un. Field Theory p. 17, 1978.