Observer-Based Robust Tracking Control Scheme with Preview Action for Uncertain Discrete-Time Systems

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This paper deals with a design problem of an observer-based robust preview control system for uncertain discrete-time systems. In this approach, we adopt 2-stage design scheme and we derive an observer-based robust controller with integral and preview actions such that a disturbance attenuation level or a given performance index is satisfactorily small for allowable uncertainties. In this paper, we show that sufficient conditions for the existence of the observer-based robust preview controller are given in terms of linear matrix inequalities (LMIs). Finally, illustrative examples are included.

Key words: preview tracking control system, observer-based control, disturbance attenuation level, 2-stage design, LMIs.

1. Introduction

One of the fundamental control problems is tracking control systems, where output signal has to track a reference signal or desired output without steady-state error. Therefore a great deal of interest has been directed to the design problem of servomechanisms for linear multivariable dynamical systems (e.g. Davison 1972, Furuta and Komiya 1972). If a controlled system has uncertainties such as unmodeled dynamics, unknown parameter variations and so on, then it is required to achieve robust stability and tracking performance. Therefore design problems of robust tracking control systems for uncertain dynamical systems have been extensively studied (e.g. Schmitendorf and Barmish 1986, Hopp and Schmitendorf 1990). It is usually assumed that desired outputs are modelled by the outputs of some free time-invariant linear systems, whose future values are not available. Thus any servomechanism has to utilize instantaneous values of desired outputs, errors and so on for control purpose.

By the way, it is well known that when the future information about reference signals and/or disturbances is available, the performance of transient responses will be greatly improved. This kind of control problem, in which information on future is utilized, is called the preview control problem (Tomizuka 1975) and a large number of design method of preview control systems have been proposed (e.g. Katayama et al. 1985, Fujisaki and Narasaki 1997). Furthermore, some $H^\infty$ preview control systems (e.g. Cohen and Shaked 1997) and robust preview tracking controllers (e.g. Takaba 1998) have been derived. Cohen and Shaked (1997) considered the problem of $H^\infty$ preview tracking control for linear time-varying systems. Also, Takaba (1998) derived a design method of a state feedback controller with integral and preview actions in terms of linear matrix inequalities (LMIs) for discrete-time systems with polytopic uncertainties.

On the other hand in most practical situations, complete state information for general multivariable dynamical systems cannot be utilized. Thus in the case that the full state information of multivariable dynamical systems cannot be measured, some observer-based quadratic stabilizing controllers (e.g. Petersen 1985, Jabbari and Schmitendorf 1993), robust $H^\infty$ controllers (e.g. Iwasaki and Skelton 1994, Park and Bien 1994) and robust output feedback control systems (e.g. Benton 1999) have been presented. Furthermore, a design method of observer-based guaranteed cost controllers for uncertain linear dynamical systems has been suggested (Oya et al. 2004). However, so far the design problem of observer-based robust preview tracking controllers for uncertain discrete-time systems has little been considered as far as we know.

From this viewpoint in this paper, we deal with an observer-based robust tracking control problem for un-
certain discrete-time systems under the assumption that finite future values of reference signals or desired output are available at each time instant. In order to derive a simple design method of the observer-based robust preview controller via LMIs framework, we adopt a similar way to the design approach developed by Oya et al. (2004). Namely, the proposed design method is roughly separated into two parts. Firstly, an observer gain matrix is designed and next, a control gain matrix is determined such that a disturbance attenuation level or an upper bound on a given performance index for an augmented difference system consisting of a tracking error, an observer, an estimation error system and the reference signal is satisfactorily small for allowable uncertainties. In this paper, we show that sufficient conditions for the existence of the observer-based robust preview tracking controller for uncertain discrete-time systems are given in terms of LMIs.

This paper is organized as follows. In Sec. 2, we define the class of uncertain discrete-time systems under consideration, and introduce an observer, an estimation error system, a tracking error system and an augmented system. Sec. 3 contains the main results. Finally, numerical examples are presented to illustrate the results developed in this paper.

In the sequel, we use the following notation. The transpose of a matrix \( \mathcal{A} \) and the inverse of one are denoted by \( \mathcal{A}^T \) and \( \mathcal{A}^{-1} \) respectively. \( H_1(\mathcal{A}) \) means \( \mathcal{A} + \mathcal{A}^T \) and \( \text{diag}(\mathcal{A}_1, \cdots, \mathcal{A}_N) \) denotes a block diagonal matrix composed of matrices \( \mathcal{A}_i \) for \( i=1, \cdots, N \). Also, \( I_n \) represents \( n \)-dimensional identity matrix. For real symmetric matrices \( \mathcal{A} \) and \( \mathcal{B} \), \( \mathcal{A} \succeq \mathcal{B} \) (resp. \( \mathcal{A} \preceq \mathcal{B} \)) means that \( \mathcal{A} - \mathcal{B} \) is positive (resp. nonnegative) definite matrix. \( E\{\cdot\} \) and Tr\{\cdot\} denote its expectation and its trace, respectively. Furthermore, \( L_{2}[0,\infty) \) is \( L_2 \)-space (i.e. the collection of all square integrable functions) and for a signal \( f(t) \in L_{2}[0,\infty) \), \( \| f(t) \|_{L_2} \) denotes its \( L_2 \) norm.

### 2. Problem Formulation

We consider an uncertain discrete-time system with the following state space representation.

\[
\begin{align*}
    x(t+1) &= A(\theta)x(t) + B(\theta)u(t) \\
    y(t) &= C(\theta)x(t)
\end{align*}
\]

where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \) and \( y(t) \in \mathbb{R}^q \) are the vectors of the state, the control input and the measured output, respectively and the parameter \( \theta \in \mathbb{R}^N \) (\( \theta=(\theta_1, \cdots, \theta_N)^T \)) is a constant vector of uncertainties. Also the matrices \( A(\theta), B(\theta) \) and \( C(\theta) \) in eq. (1) depend affinely on the parameters \( \theta_k \) for \( k=1, \cdots, N \). That is

\[
A(\theta) = \begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + \sum_{k=1}^{N} \theta_k \begin{bmatrix} A_k & B_k \\ C_k & 0 \end{bmatrix}
\]

(2)

where the matrices \( A, B \) and \( C \) denote the known nominal values and the matrices \( A_k, B_k \) and \( C_k \) for \( i=1, \cdots, N \) represent the structure of the uncertainties. In eq. (2), the unknown parameter \( \theta_k \) for \( k=1, \cdots, N \) ranges between known extremal values \( \theta^-_k \leq 0 \) and \( \theta^+_k \geq 0 \) (i.e. \( \theta_k \in [\theta^-_k, \theta^+_k] \)). This assumption means that the parameter \( \theta \in \mathbb{R}^N \) belongs to the following parameter box (Gahinet et al. 1996).

\[
\Delta =\{ \theta \in \mathbb{R}^N \mid \theta_k \in [\theta^-_k, \theta^+_k] \mbox{ for } k=1, \cdots, N \}
\]

(3)

We also assume that for \( \forall \theta \in \Delta \) the pairs \( (A(\theta), B(\theta)) \) and \( (C(\theta), A(\theta)) \) are controllable and observable, respectively and the following relation holds.

\[
\text{rank} \begin{bmatrix} A(\theta)-I_n & B(\theta) \\ C(\theta) & 0 \end{bmatrix} = n+l
\]

(4)

The assumption of eq. (4) guarantees that the system \( (A(\theta), B(\theta), C(\theta)) \) for \( \forall \theta \in \Delta \) has no zeros at \( z=1 \) and it is necessary for the existence of robustly tracking controller with integral action.

Let \( r(t) \in \mathbb{R}^l \) be the reference signal or the desired output which is assumed to be measurable. That is, we assume that the \( h \) future values of the reference signal \( r(t) \) (i.e. \( r(t+1), \cdots, r(t+h) \)) are available at each time \( t \) as well as the present and the past values of the reference signal and

\[
r(t) \in L_{2}[0,\infty)
\]

(5)

In eq. (5), \( r(t) \) is a difference vector given by \( r(t) \triangleq r(t+1) - r(t) \). Under the assumption of eq. (5), there exists a constant value \( r_\infty \) such that

\[
\lim_{t \to \infty} r(t) = r_\infty
\]

(6)

Now in order to estimate the state \( x(t) \) for the uncertain system of eq. (1), we introduce the following full state observer (Hagino and Komoriya 1989).

\[
x_c(t+1) = Ax_c(t) + Bu(t) + H(t)y(t) - Cx(t)
\]

(7)
where $H_r \in \mathbb{R}^{n \times n}$ is the observer gain matrix. In addition to the observer of eq. (7), we introduce the estimation error vector $z_e(t) = x(t) - x_r(t)$, then we see from eqs. (1) and (7) that the estimation error satisfies the relation of eq. (8). In eq. (8), $A_r(\theta) = A(\theta) - A$, $B_r(\theta) = B(\theta) - B$ and $C_r(\theta) = C(\theta) - C$, respectively.

Let $\Delta x(t) = x(t+1) - x(t)$ and $\Delta u(t) = u(t+1) - u(t)$ be the difference state vector and the difference control input vector, respectively. Additionally, we introduce the tracking error vector $e(t) = r(t) - y(t)$ and difference vectors $\Delta x(t) = x(t+1) - x(t)$, $\Delta u(t) = u(t+1) - u(t)$ and $\Delta w(t) = w(t+1) - w(t)$. Since $\Theta \in \mathbb{R}^N$ is assumed to be constant, we see from eqs. (1), (7) and (8) that the following relation is satisfied.

$$e(t+1) = e(t) - C(\theta)\Delta x(t) - C(\theta)\Delta x_e(t) + r(t)$$

(9)

Note that it follows from $u(t) \in \mathbb{L}[0, \infty)$ that the difference vector $\Delta u(t)$ is a signal of $\mathbb{L}_2$-space, i.e. $\Delta u(t) \in \mathcal{L}[0, \infty)$. Also since $h$ future values of the reference signal $r(t+1), \cdots, r(t+h)$ are available at time $t$, we define the difference vector of the reference signal described by

$$r_\delta(t) = (r^T(t), r^T(t+1), \cdots, r^T(t+h))$$

(10)

From eq. (10), the difference vector $r_\delta(t)$ satisfies

$$r_\delta(t+1) = A_\delta \Delta r_\delta(t) + B_\delta \Delta r_\delta(t+1)$$

(11)

where $A_\delta$ and $B_\delta$ are the matrices expressed as

$$A_\delta = \begin{pmatrix} 0 & I \cr 0 & 0 & \cdots & 0 \
0 & \cdots & I \cr O & 0 & \cdots & I \end{pmatrix}$$

$$B_\delta = (0 \cdots 0)$$

(12)

Furthermore from the denition of the difference vector $r_\delta(t)$, $r(t)$ can be written as

$$r(t) = \Gamma_r r_\delta(t)$$

(13)

where $\Gamma_r$ is the matrix given by

$$\Gamma_r = (I_1 \cdots 0)$$

(14)

Now we introduce an augmented vector $\xi(t) \in \mathbb{R}^{n+2N}$ given by $\xi(t) = (e^T(t), x^T(t), r^T_\delta(t), \Delta w^T(t), \Delta z_e^T(t))$, where $N_h = n + l + h + 2$. Then we obtain

$$\xi(t+1) = \mathcal{F}(\theta) \xi(t) + \mathcal{G}(\theta) u(t) + \mathcal{E} \omega(t)$$

(15)

In eq. (15), $\omega(t) \in \mathbb{R}^{n+h+1}$ and $\mathcal{F}(\theta)$, $\mathcal{G}(\theta)$ and $\mathcal{E}$ are the matrices described as

$$\mathcal{F}(\theta) = \begin{bmatrix} \mathcal{F}_{11}(\theta) & \mathcal{F}_{12}(\theta) \\
\mathcal{F}_{21}(\theta) & \mathcal{F}_{22}(\theta) \end{bmatrix}$$

(16)

$$\mathcal{F}_{11}(\theta) = \begin{bmatrix} I_1 & -C(\theta) & \Gamma_r \\
0 & A + H_r C_r(\theta) & 0 \\
0 & 0 & A_{\delta} \end{bmatrix}$$

$$\mathcal{F}_{21}(\theta) = A_r(\theta) - H_r C_r(\theta)$$

$$\mathcal{F}_{22}(\theta) = A_r(\theta) - H_r C_r(\theta)$$

$$\mathcal{G}(\theta) = (\mathcal{G}_1^T \mathcal{G}_2^T)^T$$

$$\mathcal{E}_1 = (0 B^T 0)^T$$

(17)

Note that from eqs. (16) and (17), we find that the robust stabilizability of the augmented system of eq. (15) does not depend on the preview length $h$, because all the eigenvalue of the matrix $A_{\delta}$ is stable and $r_\delta(t)$ is uncontrollable by the difference control $\Delta u(t)$.

It is well known that the integral action of the controller is introduced by including the difference control in the performance index (Katayama 1985). Therefore we define the following performance index.

$$J = \sum_{t=0}^{\infty} \{ x_e^T(t) \partial_x x_e(t) + x_e^T(t) \partial_x x_e(t) + \Delta w_T(t) \Delta w(t) \}$$

(18)

where the weighting matrices $\partial_x \in \mathbb{R}^{n \times N_x}$, $\partial_w \in \mathbb{R}^{n \times n}$ and $\partial_r \in \mathbb{R}^{n \times n}$ are positive definite which can be adjusted by designers, $x_e(t)$ is the vector given by $x_e(t) = (e^T(t), x^T(t), r^T_\delta(t), \Delta w^T(t), \Delta z_e^T(t))$. It should be noted that the physical interpretation of the performance index $J$ is to achieve the asymptotic tracking without excessive rate of change in the control input.

Using the augmented vector $\xi(t)$ and the difference control $\Delta u(t)$, we introduce the following vector.

$$\xi(t) = L \xi(t) + M \Delta u(t)$$

(19)
where $\mathcal{L}$ and $\mathcal{M}$ are the matrices given by

$$
\mathcal{L} = \begin{pmatrix}
0 & \mathcal{L}_x^{-1/2} & 0 & 0 \\
0 & 0 & \mathcal{L}_x^{-1/2} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

(20)

$$
\mathcal{M} = (0 \ 0 \ \mathcal{F}_{r1}^{T})^T
$$

Then the performance index of eq. (18) is expressed as

$$
\mathcal{J} = \sum_{t=0}^{\infty} \xi^T(t)\xi(t) = \|\xi(t)\|_{\xi^r}^2
$$

(21)

Now we consider the control law given by

$$
\bar{u}(t) = -K_x \xi(t) - K_e \sum_{j=0}^{l-1} e(j) - \sum_{j=0}^{l-1} K_{rj} \bar{r}(t+j)
$$

(22)

where $K_x$ is the control gain matrix given by $K_x = (K_x, K_r, K_e)$. From the definition of difference signals and the augmented state $x_e(t)$, we can obtain the input control $u(t)$ of eq. (23). Here, we have used the assumption that $K_x = (K_x(0), K_x(1), \cdots, K_x(h))$ and $x_e(t) = 0$, $u(t) = 0$ for any $j < 0$.

$$
u_e(t) = -K_x \xi_e(t) - K_e \sum_{j=0}^{l-1} e(j) - \sum_{j=0}^{l-1} K_{rj} \bar{r}(t+j)
$$

(23)

Substituting eq. (22) into eq. (15) yields

$$
\dot{\xi}(t+1) = \mathcal{F}_e(\theta) \xi(t) + \mathcal{G} \omega(t)
$$

(24)

where $\mathcal{F}_e(\theta)$ and $\mathcal{G}$ are the matrices such that

$$
\mathcal{F}_e(\theta) = \begin{pmatrix}
\mathcal{F}_{11}(\theta) & \mathcal{F}_{12}(\theta) \\
\mathcal{F}_{21}(\theta) & \mathcal{F}_{22}(\theta)
\end{pmatrix}
$$

(25)

$$
\mathcal{F}_e(\theta) = \mathcal{L} + \mathcal{M} K_x
$$

From the above discussion, the design problem of the observer-based robust control problem with integral and preview actions for the uncertain discrete-time system of eq. (1) is reduced to the design problem of the observer-based robust stabilizing controller for the augmented system of eq. (15). If the augmented system of eq. (15) is robustly stabilized by a given controller, then by the definition of the augmented vector $\xi(t)$, we have $e(t) \to 0$ and $x(t) - x(t-1) \to 0$ as time $t$ goes to infinity. Note that the exogenous signal $\omega(t)$ is not available for control.

3. Design of the Observer-Based Robust Preview Tracking Controller

In this section, on the basis of the design approach derived in the work of Oya et al. (2004), we consider to design the observer-based robust controller (i.e. the design problem to determine the observer gain matrix $H_2$ and the control gain matrix $K_x$) such that the augmented system of eq. (24) is robustly stable with disturbance attenuation level $\gamma > 0$ for $\forall \theta \in \Theta$. Now, we firstly give a definition for the observer-based robust controller with disturbance attenuation level $\gamma > 0$.

**Definition 1** If the augmented system of eq. (24) is robustly stable (internally stable) and the relation

$$
\gamma \|\omega(t)\|_{\xi^r} > \|\xi(t)\|_{\xi^r}
$$

(26)

holds for zero initial condition $\xi(0) = 0$, then the augmented system of eq. (24) is robustly stable with disturbance attenuation level $\gamma > 0$.

3.1 Design of the Observer Gain Matrix

From eq. (8), the estimation error satisfies the relation of eq. (27).

$$
\lambda_e(t+1) = (A(\theta) - H_2 C(\theta)) \lambda_e(t)
$$

Now for the uncertain system of eq. (28), we introduce the Lyapunov function $\mathcal{V}_H(\lambda_e, t) \triangleq \lambda_e^T(t) \Phi_H(\lambda_e, t)$ as a Lyapunov function candidate where the matrix $\Phi_H(\lambda_e, t)$ is a symmetric positive definite. Then the first order difference $\Delta \mathcal{V}_H(\lambda_e, t) \triangleq \Phi_H(\lambda_e, t + 1) - \mathcal{V}_H(\lambda_e, t)$ along the trajectory of the system of eq. (28) satisfies

$$
\Delta \mathcal{V}_H(\lambda_e, t) \triangleq \lambda_e^T(t) \Phi_H(\lambda_e, t)
$$

(29)

$$
\Phi_H(\theta) = \mathcal{A}_H(\theta)^T \Phi_H(\theta) - \Phi_H
$$

(30)

Therefore if there exist the matrices $H_2$ and $\Phi_H$ satisfying the condition $\Gamma_e(\theta) < 0$ for $\forall \theta \in \Theta$, then the quadratic stability of the system of eq. (28) is ensured. Namely, the quadratic function $\mathcal{V}_H(\lambda_e, t)$ becomes a
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Lyapunov function for the system of eq. (28). However, by introducing a symmetric positive definite matrix \( \mathcal{W}_\lambda \in \mathbb{R}_+^{n_N \times N} \) and a design parameter \( \mathcal{H}_G \in \mathbb{R}_+^{n_N \times n_N} \) which is a symmetric positive definite matrix, we now consider the condition of eq. (31) (see Remark 1).

The condition of eq. (31) can be written as eq. (32) at the top of the previous page. In eq. (32), \( \mathcal{Y}_\lambda \) is a matrix satisfying \( \mathcal{Y}_\lambda \triangleq \mathcal{Y}_\lambda H_\lambda \) and \( \mathcal{T}_\lambda \) and \( \mathcal{W}_\lambda \) are the positive definite matrices given by

\[
\mathcal{Y}_\lambda^{-1} = \frac{1}{\mathcal{Y}_\lambda^{-1} \mathcal{T}_\lambda^{-1} \mathcal{W}_\lambda \mathcal{T}_\lambda^{-1}}
\]

We now define a set of the 2\( N \) vertices of the parameter box \( \Delta \) of eq. (3) such as

\[
\Delta_{\text{wex}} \triangleq \{ \omega \in \mathbb{R}_+^{N} \mid \omega_k \in \{ \theta_k, \theta_k^+ \} \quad \text{for} \quad k = 1, \cdots, N \}
\]

Furthermore using Schur complement formula (Boyd et al. 1994), the design problem of the observer gain matrix \( H_\lambda \) is reduced to the problem of finding the matrices \( \mathcal{Y}_\lambda > 0, \mathcal{U}_\lambda > 0 \) and \( \mathcal{V}_\lambda \) which satisfy the LMI condition of eq. (35). Thus, if the solution of LMI of eq. (35) exists, then using the solution \( \mathcal{Y}_\lambda, \mathcal{U}_\lambda, \mathcal{V}_\lambda \), the observer gain matrix \( H_\lambda \) can be obtained as

\[
H_\lambda = \mathcal{Y}_\lambda^{-1} \mathcal{V}_\lambda
\]

3.2 Design of the Control Gain Matrix

In the previous section, the observer gain matrix \( H_\lambda \) has been derived. Hence, we consider to design the control gain matrix \( K_\lambda \). Firstly, we shall give a theorem for robust stability with disturbance attenuation level \( \gamma \) for the augmented system of eq. (24).

**Theorem 1** The augmented system of eq. (24) is robustly stable with disturbance attenuation level \( \gamma > 0 \) if there exist the control gain matrix \( K_\lambda \) and symmetric positive definite matrix \( \mathcal{X}_\lambda \in \mathbb{R}_+^{\{n_N+N|\omega|+N\}} \) satisfying the inequality of eq. (37)

\[
\mathcal{H}_\lambda = \mathcal{Y}_\lambda^{-1} \mathcal{V}_\lambda
\]

**Proof:** We introduce the following quadratic function

\[
\mathcal{H}_\lambda(\xi, t) \triangleq \mathcal{V}_\lambda^T(\xi, t)\mathcal{X}_\lambda(\xi, t)\mathcal{V}_\lambda(\xi, t)
\]

as a Lyapunov function candidate. By evaluating the first order difference of the function \( \mathcal{H}_\lambda(\xi, t) \), i.e., \( \Delta \mathcal{H}_\lambda(\xi, t) \triangleq \mathcal{Y}_\lambda^T(\xi, t+1) - \mathcal{Y}_\lambda^T(\xi, t) \) along the trajectory of the augmented system of eq. (24) and considering the Hamiltonian

\[
\mathcal{H}(\xi, \omega) \triangleq \Delta \mathcal{H}_\lambda(\xi, t) + \mathcal{V}_\lambda(\xi, t-1) - \mathcal{V}_\lambda(\xi, t) + \mathcal{V}_\lambda^T(\xi, t)\gamma^2\omega^T(\xi, t-1)\omega(t)
\]
we get the relation of eq. (39). If the inequality condition of eq. (37) is satisfied, then we easily see that the inequality condition of eq. (37) is equivalent to

\[ \mathcal{H}(\xi, \omega) < 0 \quad \text{for} \quad \forall \theta \in \Delta \quad (40) \]

because the relation of eq. (39) can be written as

\[ \mathcal{H}(\xi, \omega) = \begin{bmatrix} \xi(t) \\ \omega(t) \end{bmatrix}^T \Phi_2(\theta) \begin{bmatrix} \xi(t) \\ \omega(t) \end{bmatrix} \quad (41) \]

Thus we easily see from (1,1)-block of the inequality of eq. (37) that the augmented system of eq. (24) is robustly stable (internally stable). Moreover for zeroinitial condition \( \xi(0) = 0 \), summing up the inequality of eq. (40) yields the following relation.

\[ \gamma^2 \| \omega(t) \|_2 \leq \| \xi(t) \|_2, \quad (\mathcal{F}_2 = 0) \quad (42) \]

From the above discussion, if the matrices \( K_e \) and \( \bar{\mathcal{F}}_2 \) satisfying the condition of eq. (37), then the augmented system of eq. (24) is robustly stable with disturbance attenuation level \( \gamma \). Therefore it follows that the result of the theorem is true. \( \Box \)

**Theorem 1** provides a sufficient condition for the existence of the observer-based robust preview tracking controller with disturbance attenuation level \( \gamma > 0 \). The following theorem provides a design method of the observer-based robust preview tracking controller with disturbance attenuation level \( \gamma \).

**Theorem 2** Consider the augmented system of eq. (24) and the controller which is composed of the control law of eq. (23) and the observer of eq. (27).

For a given constant \( \gamma > 0 \), if there exist the solution \( \mathcal{F}_2^e > 0 \), \( \mathcal{W}_e^e \) and \( \bar{\mathcal{F}}_2 > 0 \) of the LMI of eq. (43), then the observer-based robust controller consisting of the control gain matrix \( K_e \) which can be obtained as

\[ K_e = \mathcal{W}_e \bar{\mathcal{F}}_2^{-1} \quad (44) \]

and the observer gain matrix \( H_e \) which is designed such as \( H_e = \mathcal{F}_2^e \bar{\mathcal{F}}_2 \), by solving the LMI condition of eq. (39) robustly stabilizes the augmented system of eq. (24) and achieves the disturbance attenuation level \( \gamma \). In eq. (48), the matrices \( \mathcal{F}_2^e \), \( \mathcal{W}_e^e \) and \( \bar{\mathcal{F}}_2(\theta) \bar{\mathcal{F}}_2 \) are the matrices expressed as eqs. (45) and (46).

\[ \mathcal{F}_2(\theta) = \begin{bmatrix} \mathcal{F}_{11}(\theta) \mathcal{F}_{12}(\theta) \\ \mathcal{F}_{21}(\theta) \mathcal{F}_{22}(\theta) \end{bmatrix}, \quad \mathcal{W}_e(\theta) = \begin{bmatrix} \mathcal{W}_{e1}(\theta) \mathcal{W}_{e2}(\theta) \end{bmatrix} \quad (45) \]

**Proof:** By pre- and post-multiplying eq. (37) by the matrix diag\( (\mathcal{F}_2^e I_{n+m+N}) \) and using the matrices \( \mathcal{F}_2^e \in \mathcal{H}^{(n+m+N) \times (n+m+N)} \) and \( \mathcal{W}_e \in \mathcal{H}^{(n+m+N)} \), simple algebraic manipulation gives the matrix inequality of eq. (48). Furthermore by using the Schur complement formula (Boyd et al. 1994), the inequality of eq. (48) is reduced to the following condition.

The matrix inequality condition of eq. (47) is LMI in \( \mathcal{F}_2^e \), \( \bar{\mathcal{F}}_2 \) and \( \mathcal{W}_e^e \) because the matrix \( \bar{\mathcal{F}}_2(\theta) \bar{\mathcal{F}}_2 \) is expressed as eq. (48). Namely, the matrix inequality condition of eq. (46) is equivalent to the LMI condition of eq. (43). Furthermore, we easily see from eq. (45) that if there exists the solution of LMI of eq. (43), then the control gain matrix \( K_e \) is obtained as eq. (44).

From the above discussion, the proof of **Theorem 2** is completed.

In eq. (43) by setting \( \gamma^* \equiv \gamma^2 \), the inequality condition of eq. (43) can be written as eq. (49). The condition of eq. (49) is LMI in \( \mathcal{F}_2^e \), \( \mathcal{W}_e^e \) and \( \gamma^* \) because in eq. (49), the parameter \( \gamma^* \) appears anly. The LMI condi-
tion of eq. (49) also defines a convex solution set of \((\mathcal{S}_r, \mathcal{S}_z, W_r, \gamma^*)\). Therefore various eccentric convex optimization algorithms can be used to test whether the LMI of eq. (49) is solvable and to generate particular solutions. Moreover, its solutions parametrize the set of the observer-based robust preview tracking controllers and the parametrized representation can be exploited to design the observer-based robust tracking controller with some additional requirements.

In this paper therefore, we seek to minimize \(\gamma^*\) subject to the constraint of eq. (49) and the observer gain matrix designed as eq. (36). Namely, we consider to design the control gain matrix which minimizes the parameter \(\gamma^*\) and then we get the following theorem. Note that the observer-based robust controller minimizing the parameter \(\gamma^*\) is not optimal but sub-optimal, because we consider the constrained convex optimization problem of eq. (50) under the observer gain matrix designed as \(H_r = \mathcal{Y}_r^{-1} \mathcal{Y}_\lambda\) (Oya et al. 2004).

**Theorem 3** Consider the augmented system of eq. (24) and the controller which is composed of the control law of eq. (23) and the observer of eq. (27). Then the observer-based robust control consisting of the control gain matrix \(K_x\) which is designed as \(K_x = W_x^{-1} \mathcal{S}_r^{-1}\) and the observer gain matrix \(H_r\) which is obtained as \(H_r = \mathcal{Y}_r^{-1} \mathcal{Y}_\lambda\) by solving the LMI of eq. (35) robustly stabilizes the augmented system of eq. (24) with disturbance attenuation performance \(\gamma = (\sqrt{\gamma^*})\).

**3.3 Special Case:** \(r(t+j) = r(t+h)\) for \(j \geq h+1\)

In the above, the reference signal \(r(t)\) satisfying \(r(t) = r(t+1) - r(t) \in \mathcal{L}_2[0, \infty)\) is considered. Now we consider a special case that the reference signal \(r(t)\) satisfies the relation \(r(t+j) = r(t+h)\) for \(j \geq h+1\). This control problem is also considered in the work of Takaba (1998).

From the definition of \(\omega(t)\), if the reference signal \(r(t)\) satisfies the relation \(r(t+j) = r(t+h)\) for \(j \geq h+1\), then \(\omega(t) = 0\). Namely, the augmented system of eq. (24) can be rewritten as

\[
\begin{align*}
\xi(t+1) &= \mathcal{F}_K(\theta)\xi(t) \\
\zeta(t) &= \mathcal{D}_K \xi(t)
\end{align*}
\]

Also from the condition of eq. (37), we get the condition

\[
\mathcal{F}_K(\theta)\mathcal{F}_z \mathcal{F}_K^T(\theta) - \mathcal{D}_K \mathcal{D}_L + M \mathcal{W}_z^T < 0 \quad \text{for } \forall \theta \in \Delta
\]

and for all initial condition \(\xi(0)\), we have the following relation, because \(\omega(t) = 0\):

\[
\mathcal{F}_z < e^{\mathcal{F}_z(\theta)T} \xi(0)
\]

Furthermore, by introducing the complementary matrices \(\mathcal{L}_z = \mathcal{D}_K \mathcal{F}_z, \mathcal{W}_z = \mathcal{F}_z^{-1}, \mathcal{W}_z = \mathcal{D}_z K_d \mathcal{L}_z\), and using the Schur complement formula (Boyd et al. 1994), we find that the condition of eq. (52) is equivalent to the condition of eq. (54).

In this case, we adopt a similar way to observer-based guaranteed cost control (Oya et al. 2004). Namely, we assume that the initial value \(\xi(0)\) is zero mean random vector satisfying \(E\{\xi(0)\xi^T(0)\} = I_{n+N_0}\) and \(E\{\xi(0)\} = 0\). Then the upper bound on the performance index of eq. (53) is given as \(Tr\{\mathcal{X}_z\}\). Thus, we consider the constrained optimization problem.

However, the condition of eq. (54) is LMIs in \(\mathcal{X}_z, \mathcal{L}_z\) and \(\mathcal{W}_z\). Thus we introduce a complementary variable \(\mathcal{X}_z \in \mathcal{R}^{(n+N_0)(n+N_0)}\) satisfying

\[
\begin{pmatrix}
-\mathcal{F}_z & 0 & \mathcal{F}_K(\theta)\mathcal{L}_z & 0 \\
\mathcal{L}_z^T + \mathcal{W}_z^T \mathcal{M}^T & -\mathcal{F}_z & 0 & 0 \\
\mathcal{F}_z^T \mathcal{F}_K^T(\theta) & -\mathcal{F}_z & 0 & -\mathcal{F}_z \epsilon \\
0 & 0 & 0 & -\gamma^* I_{n+m+N_0}
\end{pmatrix}
\]
Then the minimization problem of $\text{Tr}\{\mathcal{X}_z\}$ can be transformed into that of $\text{Tr}\{\mathcal{X}_z\}$. Therefore, the design problem of the control gain matrix to minimize the upper bound on the performance index of eq. (53) is reduced to the following constrained convex optimization problem, because the condition of eq. (56) is also LMIs in $\mathcal{X}_z$ and $\mathcal{Y}_z$.

Note that the observer-based robust controller minimizing the upper bound on the performance index of eq. (53) is not optimal but sub-optimal, because we consider the optimization problem of eq. (57) under the observer gain matrix designed as $H_r = \mathcal{Y}_{K_r}^{-1} \mathcal{Y}_{K_r}$.

As a result, the following theorem is obtained.

**Theorem 4** Consider the augmented system of eq. (51) and the controller which is composed of the control law of eq. (23) and the observer of eq. (27).

There exists the control gain matrix $K_r$ minimizing the upper bound on the performance index of eq. (53), if there exist the optimal solution $\mathcal{X}_z > 0$, $\mathcal{Y}_z > 0$ and $\mathcal{Y}_z > 0$ of the constrained convex optimization problem of eq. (57). Also, using the solution of the LMI condition of eq. (35), the observer gain matrix $H_r$ is designed in advance such as $H_r = \mathcal{Y}_{K_r}^{-1} \mathcal{Y}_{K_r}$.

If the solution $\mathcal{X}_z$, $\mathcal{W}_{z}$, $\mathcal{Y}_z$, and $\mathcal{Y}_z$ of the constrained convex optimization problem is obtained, then the control gain matrix is given by $K_r = \mathcal{W}_{z} \mathcal{Y}_{z}^{-1}$.

**Remark 1** In this paper, the observer of eq. (7) is designed such that quadratic stability of the system of eq. (28) is ensured and the condition of eq. (31) is satisfied, because in order to get the control gain matrix $K_r$ and the symmetric positive definite matrix $\mathcal{X}_z$ satisfying the inequality of eq. (37) or eq. (54), (2, 2)-block of the matrix inequality condition $\mathcal{X}_z(\theta) \mathcal{X}_z(\theta) + \mathcal{S}_R(\theta) - \mathcal{S}_R < 0$ for $\theta \in \Delta$ (i.e. (1, 1)-block of the condition of eq. (37) or eq. (52)) must be negative definite. Namely, the observer gain matrix $H_r$ has to be determined, making allowance for the inequality of eq. (58) at the top of the previous page and that is a necessary condition for the existence of the control gain matrix $K_r$ and the symmetric positive definite $\mathcal{X}_z$ satisfying the condition of eq. (37) or eq. (54).

Thus introducing a symmetric positive definite matrix $\mathcal{W}_{\gamma} \in \mathbb{R}^{N_s \times N_s}$ and a design parameter $\gamma \in \mathbb{R}^{I}$, we consider the condition of eq. (31).

**Remark 2** If the augmented system of eq. (24) is robustly stable with disturbance attenuation level $\gamma$, then for $\omega(0) \in \mathcal{L}_{2}$, $\forall \xi(0)$ and $\forall \theta \in \Delta$ we have

$$\lim_{t \to \infty} \xi(t) = 0$$

**Remark 3** The result shown in Sec. 3.3 is equivalent to the existing result (Oya et al. 2005), i.e. the result derived by Oya et al. (2005) is included as a special case of the result of this paper. Namely, the result of this paper is an extended version of the existing result (Oya et al. 2005) and thus the resulting controllers can be applied to more practical reference signals.

### 4. Illustrative Examples

In this section, we illustrate the effectiveness of the proposed observer-based robust preview tracking controller by the following simple example.

Consider the uncertain discrete-time system with unknown parameters of eq. (60). In eq. (60), the parameters $\delta_1$ and $\delta_2$ are the uncertainties and are assumed to take the value within the interval $[-0.15, 0.15]$ and $[-0.10, 0.10]$, respectively.

$$x(t+1) = \begin{pmatrix} 1.0 & 1.0 \\ 0.5 & 0.85 + \delta_1 \end{pmatrix} x(t) + \begin{pmatrix} 1.0 \\ 1.0 \end{pmatrix} u(t)$$

$$y(t) = (1.0 + \delta_2) x(t)$$

By choosing the design parameter $\gamma = L_2$ and solving the LMI condition of eq. (28), we obtain the following observer gain matrix.

$$H_r = (1.28941, 0.81503)^T$$

Next, we select the weighting matrices $\mathcal{W}_z, \mathcal{Z}_r$ and $\mathcal{R}_r$ such that $\mathcal{Z}_r = I_2, \mathcal{Z}_r = 4.0 I_2$ and $\mathcal{R}_r = 1.0$ and let $h = 5$. By applying **Theorem 3** and solving the constrained convex optimization problem of eq. (50), we get the control gain matrix $K_r$ given by eq. (62) and the parameter...
Now we assume $x_e(j) = 0$, $u(j) = 0$, and $r(j) = 0$ for any $j < 0$ mentioned above. Also in this example, the reference signal $r(t)$ is supposed to vary such that

$$r(t) = \begin{cases} 0.0 & \text{for } t < 20 \\ 10.0 & \text{for } t \geq 20 \end{cases}$$  \hspace{1cm} (63)

In order to examine the robustness of the proposed controller, we consider four cases such that

- **Case 1**: $\delta_1 = -0.15$ and $\delta_2 = -0.1$
- **Case 2**: $\delta_1 = -0.15$ and $\delta_2 = 0.1$
- **Case 3**: $\delta_1 = 0.15$ and $\delta_2 = -0.1$
- **Case 4**: $\delta_1 = 0.15$ and $\delta_2 = 0.1$

Namely, the unknown parameters $\delta_1$ and $\delta_2$ take the values of extremal point of the intervals $[-0.15, 0.15]$ and $[-0.1, 0.1]$, respectively.

The results of the simulation of this example are depicted in Figure 1–5. We see from these figures that the proposed observer-based robust preview controller achieves robust tracking performance and robustly stabilizes the augmented system of eq. (24).

### 5. Conclusions

In this paper, a design method of an observer-based robust preview tracking control system for uncertain discrete-time systems under the assumption that finite future values of reference signals are available at each
time instant has been presented. The proposed observer-based robust preview tracking controller is easily obtained through a constrained convex optimization problem, because adopting 2-stage design approach (Oya et al. 2004), the design problem of the observer-based robust tracking controller with preview action is reduced to the LMIs. Therefore, the proposed observer-based robust controller with integral and preview actions can be easily obtained by using commercially available software such as MATLAB’s LMI Control Toolbox and Scilab’s LMITOOL. In this paper, the reference signal $r(t) \in L_2[0, \infty)$ is considered and the observer-based robust preview tracking controller is determined such that the augmented system is robustly stable with disturbance attenuation level $\gamma$. Moreover, the special case that the reference signal satisfies the relation $r(t+j) = r(t+h+1)$ (i.e. $\|w(t)\| \leq \|r(t+h+1)\|$) are discussed. Namely, the existing result (Oya et al. 2005) is included as a special case of the result of this paper and therefore the resulting controllers can be applied to more practical reference signals.

The future research subject is an extension of the proposed design method of an observer-based robust preview tracking control system to such a broad class of systems as uncertain time-delay systems, uncertain large-scale interconnected systems and so on. Furthermore in future work, we will examine the controller design algorithm for the minimization of the disturbance attenuation level $\gamma$ or the upper bound on the performance index for special case mentioned in Sec. 3.3, because the observer-based robust tracking controller derived by our design method is not optimal.

References