SHONAN INSTITUTE OF
TECHNOLOGY JOURNAL
Vol. 53, No. 1, 2019

Discontinuous solutions of Euler equations in the plane

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Abstract:
We prove that a stationary solution of vortex sheet equations is a circle if and only if a vortex sheet is a smooth simple closed curve, and investigate the stability of this stationary solution. In addition, we prove finite time analyticity of the nonlinear nonstationary problem of a vortex sheet which is close to a circle.

KEY WORDS : vortex sheet, stationary solution, linear stability, nonstationary solution

Introduction

We consider the Euler equations for an incompressible ideal fluid for $t \in (0, \infty)$ in the plane

(1) $$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p$$

(2) $$\operatorname{div} u = 0$$

where $u = (u_1(x,t), u_2(x,t))$ is the fluid velocity and $p = p(x,t)$ is the scalar pressure.

We are concerned with the motion of vortex sheets of the Euler equations, i.e., an irrotational flow is discontinuous across a curve, i.e., vortex sheet $\Gamma(t) = \{x(\lambda, t) \in R^2 \mid \lambda \in R\}$; hence, the vorticity $\nabla^{\perp} u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$ is concentrated on it. Then the vorticity density $\Omega = \Omega(\lambda, t)$ is defined by

(3) $$\iint_{\mathrm{R}^2} u(x,t) \cdot \nabla^{\perp} f(x) dx = \int_{\mathrm{R}} \Omega(\lambda, t) f(x(\lambda, t)) d\lambda$$

for any $f \in C_0^{\infty}(\mathrm{R}^2)$, where $\nabla^{\perp} f = \left(\frac{\partial f}{\partial x_2}, -\frac{\partial f}{\partial x_1} \right)$.

The system that governs the evolution of a vortex sheet and a vorticity density on it is derived from the Euler equations (1), (2) with the definition of the vorticity (3), established in [5]:

(4) $$[u] \cdot \left(\frac{\partial x}{\partial \lambda} \right)^{\perp} = 0$$

(5) $$\left(\frac{\partial x}{\partial t} - V \right) \cdot \left(\frac{\partial x}{\partial \lambda} \right)^{\perp} = 0$$

(6) $$\frac{\partial \Omega}{\partial t} + \frac{\partial}{\partial \lambda} \left\{ \frac{\Omega}{\left| \frac{\partial x}{\partial \lambda} \right|^2} \left(V - \frac{\partial x}{\partial t} \right) \cdot \frac{\partial x}{\partial \lambda} \right\} = 0$$

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where

(7) $$\Omega = [u] \cdot \frac{\partial x}{\partial \lambda},$$

$[u]$ is the velocity jump across $\Gamma(t)$ and $V = (V_1(\lambda, t), V_2(\lambda, t))$ is the mean of the two velocities on both side of $\Gamma(t)$, and $x^{\perp} = (x_2, -x_1)$.

A simple illustration is given by

(8) $$u = \begin{cases} (\Omega_0/2, 0) & x_2 < 0 \\ (-\Omega_0/2, 0) & x_2 > 0 \end{cases}$$

where Ω_0 is a constant vorticity density on the x_1-axis. This flow is linearly unstable, known as the Kelvin-Helmholtz instability, because the amplitude of the *k*-Fourier mode of an initial disturbance of curves or vorticity densities has an exponential growth in time at the rate $|k\Omega_0|$. C. Sulem, P.L. Sulem, C. Bardos and U Frisch [5] prove the existence theorem for the initial data with finite time persistence of analyticity through an abstract Cauchy-Kowalewski theorem. J. Duchon and R. Robert [2] show one special analytic choice of the initial circulation distribution for which there is a global piecewise-analytic solution. The work of the above authors is made on the assumption that a vortex sheet is close to a straight line.

1. Stationary problem

This section is devoted to the consideration of the stationary problem induced from (5), (6):

(9) $$V \cdot \left(\frac{\partial x}{\partial \lambda}\right)^{\perp} = 0$$

(10) $$\frac{\partial}{\partial \lambda}\left\{\left([u] \cdot \frac{\partial x}{\partial \lambda} \Big/ \left|\frac{\partial x}{\partial \lambda}\right|^2\right)\left(V \cdot \frac{\partial x}{\partial \lambda}\right)\right\} = 0$$

and (4).

1.1 Stationary solution

We assume that a vortex sheet is a smooth simple closed curve. Then we have

Lemma 1. *Let* u_i *and* u_e *be the velocities inside and outside the vortex sheet* Γ, *respectively. Then,*

(11) $$u_i = 0 \quad \text{in the inside of } \Gamma,$$

and

(12) $$u_e \cdot n = 0, \quad u_e \cdot \tau = 0 \quad \text{on } \Gamma,$$

where n *is an outward normal vector and* τ *is a unit tangential vector on* Γ.

Proof. By adding (4) and (9) we have

(13) $$u_i \cdot \left(\frac{\partial x}{\partial \lambda}\right)^{\perp} = 0, \quad u_e \cdot \left(\frac{\partial x}{\partial \lambda}\right)^{\perp} = 0 \quad \text{on } \Gamma.$$

Hence we easily see that u_i vanishes since u_i is both solenoidal and irrotational. Substituting this into (10), we get the second equation of (12).

Using the above lemma, we shall rewrite the stationary problem. We note that there is a harmonic function φ such that $\nabla^{\perp}\varphi = u_e$ since u_e is solenoidal and irrotational. Then it is easy to see $u_e \cdot n = \frac{\partial \varphi}{\partial \tau}$ and $u_e \cdot \tau = \frac{\partial \varphi}{\partial n}$ on Γ. Hence, by virtue of (12), the stationary problem is reduced to finding both a smooth simple closed curve Γ and a harmonic function φ outside Γ satisfying the boundary conditions:

$$\frac{\partial \varphi}{\partial n} = c_1 \quad \textit{and} \quad \varphi = const. \qquad \textit{on } \Gamma \ , \tag{14}$$

where $c_1 \neq 0$ is the arbitrarily given constant (if $c_1 = 0$, then $\Omega = 0$).

First we easily see that the stationary problem has the following solution; if Γ is a circle about x_0 with the length of the circumference L, then φ is represented in the form

$$\varphi(x) = \frac{c_1 L}{2\pi} \log \frac{1}{|x - x_0|} + const. \tag{15}$$

With respect to this stationary problem, harmonic functions must satisfy over-determined boundary conditions. It hence seems to be a natural question whether or not there exists another solution although Γ is unknown. This uniqueness problem has an affirmative answer.

More precisely, we state

Theorem 1. *Let Γ be a simple closed curve of class $C^{1+\theta}$ for some $0 < \theta < 1$. Let D be the outside domain of Γ and let L denote the length of the perimeter of Γ.*
Suppose that there exists a harmonic function φ in D satisfying the boundary condition (14) *and*

$$\nabla \varphi \ \textit{ remains bounded in } \overline{D}. \tag{16}$$

Then Γ is a circle and φ has the specific form (15).

Remark. 1) Condition (16) seems to be a physically reasonable assumption; this means that the velocity of the flow remains bounded at infinity. 2) It is easy to see that the vorticity density is given by $\Omega = c_1 L / 2\pi$.

The following lemma is needed to prove the above theorem.

Lemma 2. *Let Γ and Γ_1 be simple closed curves of class C^1, where Γ is inside Γ_1.*
Let D_1 be a domain bounded by an outer contour Γ_1 and an inner contour Γ.
Assume that φ is a harmonic function in D_1 with $\varphi \in C(\overline{D_1})$ satisfying the following properties,

$$\frac{\partial \varphi}{\partial n} \ \textit{ has a definite sign on } \Gamma_1, \tag{17}$$

$$\nabla \varphi \ \textit{ can be extended by continuity to } \Gamma_1. \tag{18}$$

Then, for any critical point $x_0 \in D_1$ of φ, i.e., $\nabla \varphi(x_0) = 0$, the following inequalities hold.

$$\min\{\varphi(\xi) \mid \xi \in \Gamma\} < \varphi(x_0) < \max\{\varphi(\xi) \mid \xi \in \Gamma\}. \tag{19}$$

This lemma may be known, but for the convenience of the reader, we shall give the proof at the end of this subsection.

Corollary. *On the assumption of lemma 2, if $\varphi_{|\Gamma} = const.$, then $\nabla \varphi \neq 0$ in D_1.*

Proof of theorem1.

We first show that φ is represented in the form:

(20) $$\varphi(x)=\frac{c_1}{2\pi}\int_\Gamma \log\frac{1}{|x-\xi|}ds_\xi+const.$$

Let R be a positive number such that $R>\max\{|\xi|\mid \xi\in\Gamma\}$. We denote by Γ_R a circumference about the origin with radius R and by B_R a domain bounded by Γ_R. We apply Green's formula to $B_R\cap D$ for $\varphi-\varphi_{|\Gamma}$ and $\frac{1}{2\pi}\log\frac{1}{|x-\xi|}$. Using (14) and the identity: $\frac{1}{2\pi}\int_{\Gamma_R}\frac{\partial}{\partial n_\xi}\log\frac{1}{|x-\xi|}ds_\xi=-1$, we obtain

(21) $$\varphi(x)=\frac{c_1}{2\pi}\int_\Gamma \log\frac{1}{|x-\xi|}ds_\xi+\frac{1}{2\pi}\int_{\Gamma_R}\left\{\frac{\partial\varphi(\xi)}{\partial n_\xi}\log\frac{1}{|x-\xi|}-\varphi(\xi)\frac{\partial}{\partial n_\xi}\log\frac{1}{|x-\xi|}\right\}ds_\xi$$
$$\equiv\varphi_1(x)+\varphi_2(x) \qquad \textit{for any}\quad x\in B_R\cap D.$$

We know that φ_2 (single layer potential and double layer potential) is a harmonic function in B_R and that φ and φ_1 are harmonic in D. In addition, the identity $\varphi_2=\varphi-\varphi_1$ holds in $B_R\cap D$. Hence there is a harmonic extension $\tilde\varphi_2$ in $B_R\cup D=R^2$ such that $\tilde\varphi_2=\varphi-\varphi_1$ in D.

Since $\nabla\varphi_1(x)=O\left(|x|^{-1}\right)$ as $|x|\to\infty$ and (16), $\frac{\partial\tilde\varphi_2}{\partial x_j}$ (j=1, 2) is a bounded harmonic function in on R^2. This yields that $\frac{\partial\tilde\varphi_2}{\partial x_j}$ (j=1, 2) is identically constant, and hence we see that $\tilde\varphi_2(x)=b\cdot x+const.$, where b is an arbitrary constant vector. From the property of the single layer potential, the equation: $\frac{\partial\varphi_1}{\partial n}_{|\Gamma}=c_1$ holds. This, together with (14), yields $\frac{\partial\tilde\varphi_2}{\partial n}_{|\Gamma}=b\cdot n_{|\Gamma}=0$, and therefore $b=0$. Hence we obtain (20).

We next show that φ satisfies the identity

(22) $$|\nabla\varphi(x)|=|c_1|\exp\left(-\frac{c_2-\varphi(x)}{c_1r_0}\right) \qquad \textit{for any}\quad x\in D.$$

where $c_2=\varphi_{|\Gamma}\ (\equiv const.)$ and $r_0=L/2\pi$. To this end we begin with proving

(23) $$\nabla\varphi(x)\neq 0 \qquad \textit{for any}\quad x\in D.$$

Differentiating (20) we have

(24) $$\nabla\varphi(x)=-\frac{c_1}{2\pi}\int_\Gamma\frac{x-\xi}{|x-\xi|^2}ds_\xi$$

From this, for any $x \in D \setminus B_R$

$$\left| \frac{x}{|x|} \cdot \nabla \varphi(x) \right| \geq \frac{|c_1|}{2\pi} \left(R - \max_{\xi \in \Gamma} |\xi| \right) \int_\Gamma \frac{1}{|x-\xi|^2} ds_\xi > 0$$

This shows $\nabla \varphi(x) \neq 0$ in $D \setminus B_R$, and in particular, $\frac{\partial \varphi}{\partial n}_{|\Gamma_R} = \frac{x}{|x|} \cdot \nabla \varphi(x)_{||x|=R} \neq 0$.

Furthermore, since $\varphi_{|\Gamma} = const.$ holds, applying Corollary of lemma 2 to φ in $D \cap B_R$ we have $\nabla \varphi(x) \neq 0$ in $D \cap B_R$. Hence we obtain (23).

Since φ is a harmonic function with (23), it is easy to see

(25) $$\log|\nabla \varphi(x)| \quad \textit{is a harmonic function in } D.$$

In addition, since the boundary condition (14) yields $\nabla \varphi = \frac{\partial \varphi}{\partial n} n = c_1 n$ on Γ, $\log|\nabla \varphi(x)|$ satisfies

(26) $$\log|\nabla \varphi(x)| = \log|c_1| \quad on \ \Gamma.$$

From (24) and identity: $\left| \frac{x-\xi}{|x-\xi|^2} - \frac{x}{|x|^2} \right| = \frac{|\xi|}{|x||x-\xi|}$ we see

$$\nabla \varphi(x) = -c_1 r_0 \frac{x}{|x|^2} + O\left(|x|^{-2}\right) \quad as \quad |x| \to \infty.$$

This implies

(27) $$\lim_{|x| \to \infty} \left(\log|\nabla \varphi(x)| - \log \frac{1}{|x|} \right) = \log|c_1| r_0.$$

On the other hand, it follows from (20) that

(28) $$\lim_{|x| \to \infty} \left(\varphi(x) - c_1 r_0 \log \frac{1}{|x|} \right) = const..$$

Consequently, setting $\psi(x) = \varphi(x) - c_1 r_0 \log|\nabla \varphi(x)|$ and combining the above results (25)~(28) with the assumption for φ we see that ψ is a harmonic function in D satisfying $\psi(x) = c_2 - c_1 r_0 \log|c_1|$ on Γ and ψ remains bounded at infinity. Hence from the uniqueness theorem for exterior Dirichlet problem we obtain $\psi(x) = c_2 - c_1 r_0 \log|c_1|$ in D, thus showing (22).

Let $\Psi(x) = \exp \frac{2(c_2 - \varphi(x))}{c_1 r_0}$. Then differentiating $\Psi(x)$ and using (22) we have

(29) $$|\nabla \Psi((x)|^2 = \frac{4}{{r_0}^2} \exp \frac{2(c_2 - \varphi(x))}{c_1 r_0} = \frac{4}{{r_0}^2} \Psi(x)$$

Differentiating (29) in x_j ($j = 1,2$) yields

(30) $$\nabla \Psi(x) \cdot \nabla \frac{\partial}{\partial x_j} \left\{ \Psi(x) - \frac{|x|^2}{{r_0}^2} \right\} = 0$$

Since it follows straightforward computation that $\Psi(x)-|x|^2/r_0{}^2$ is harmonic in D and (22), we also have

(31) $$\nabla\Psi(x)^{\perp}\cdot\nabla\frac{\partial}{\partial x_j}\left\{\Psi(x)-\frac{|x|^2}{r_0{}^2}\right\}=0$$

Hence noting from (29) that $\nabla\Psi(x)\neq 0$ for any $x\in D$ we have

$$\nabla\frac{\partial}{\partial x_j}\left\{\Psi(x)-\frac{|x|^2}{r_0{}^2}\right\}=0 \quad (j=1,2) \quad \text{in} \quad D.$$

We thus see that $\Psi(x)-|x|^2/r_0{}^2$ is a linear function, and moreover since $\Psi(x)$ satisfies (29), we obtain

(32) $$\Psi(x)-\frac{|x|^2}{r_0{}^2}=\frac{1}{r_0{}^2}\left(-2x_0\cdot x+|x_0|^2\right),$$

where x_0 is an arbitrary point in R^2. Hence we have

$$\exp\frac{2(c_2-\varphi(x))}{c_1 r_0}=\Psi(x)=\frac{1}{r_0{}^2}|x-x_0|^2$$

Noting $\varphi_{|\Gamma}=c_2$ we see that $|x-x_0|^2=r_0{}^2$ for any $x\in\Gamma$. Substituting this into (20) we obtain (15).

Remark. The above proof is inspired by [4] and [6].

Proof of lemma 2.

We shall show that (19) holds for the case $\dfrac{\partial\varphi}{\partial n}_{|\Gamma_1}>0$.

We begin with investigating the behavior of trajectories satisfying the gradient system:

(33) $$\frac{dx}{dt}=\nabla\varphi(x).$$

As is known the theory of ordinary differential equations, noncontinuable solutions of (33) have the following properties

a) *Trajectories cannot intersect each other at any regular point of* φ.

b) *Both endpoints of a trajectory are certainly located on the boundary of* D_1 *or at some critical point of* φ *in* D_1.

c) $\varphi(x(t))$ *is a strictly monotone increasing function in* t.

We denote by S the set of all critical points of φ in D_1. Then S is a finite set since it follows from $\triangle\varphi=0$ that the critical points are isolated. To study the behavior of trajectories in the neighborhood of a critical point we expand the right-hand side of (33) into Taylor series at each point $a\in S$. Using $\triangle\varphi=0$ and introducing polar coordinates $(r=|x-a|,\theta)$ we can rewrite (33) in the neighborhood of a as follows. There exists $n\geq 2$ for each $a\in S$ such that

(34)
$$\frac{dr}{dt} = cr^{n-1}\cos(n\theta - \theta_0) + O(r^n)$$
$$\frac{d\theta}{dt} = -cr^{n-2}\sin(n\theta - \theta_0) + O(r^{n-1})$$

where the constants c $(c \neq 0)$ and θ_0 depend only on the values of n-th derivatives of φ and a.

The above equations read that behavior of trajectories in the neighborhood of a degenerate critical point, i.e., the case $n \geq 3$, is similar to that in the neighborhood of a saddle point corresponding to the case $n = 2$; more precisely,

(35) *There exist exactly n stable branches and exactly n unstable branches and alternately for each critical point.*

All stable branches approach the corresponding critical points from another endpoints as t increases. On the other hand, the sign of the outer normal derivative of φ on Γ_1 is positive, hence the following holds

(36) *Another endpoint of any stable branch cannot be located on* Γ_1.

Let a_m be the critical point satisfying $\varphi(a_m) = \min\{\varphi(a) \mid a \in S\}$ and let $\gamma(t)$ be a stable branch of a_m. Then c) implies that $\gamma(t)$ cannot approach any other critical point in D_1 with decreasing t. Hence it follows from (36) and b) that $\gamma(t)$ approaches Γ and $\varphi(\gamma(t))$ decreases with decreasing t. This yields the first inequality (19).

Let a_M be the critical point satisfying $\varphi(a_M) = \max\{\varphi(a) \mid a \in S\}$. To show the second inequality (19), by c) it is sufficient to prove the existence of an unstable branch of a_M whose another endpoint is located on Γ.

Assume that there is no unstable branch of a_M which reaches Γ. Then since c) implies that any unstable branch of a_M cannot approach any other critical point, we deduce that all unstable branches of a_M reach the outer contour Γ_1. Hence the continuous curve formed by two of these branches and the point a_M divided D_1 into two subdomains; one subdomain lies on the same side as Γ and another subdomain lies on the opposite side of Γ. Denote by S' the set of the point a_M and all critical points in the latter subdomain and let $a'_m \in S'$ be the point satisfying $\varphi(a'_m) = \min\{\varphi(a) \mid a \in S'\}$. Then, if a'_m has a stable branch, then in the same arguments as the proof of the first inequality (19), we note another endpoint of this branch in nowhere to be found. As a result, we deduce that a'_m has no stable branch. This contradicts (35).

Hence we conclude that a_M has an unstable branch reaching Γ. This implies the second inequality (19).

Remark. To prove (19) in the case $\frac{\partial \varphi}{\partial n}_{|\Gamma_1} < 0$, it is sufficient to change the sign of the time t; we may omit the detail.

1.2 Linear stability of stationary solution

In this subsection we shall analyze linear stability of the stationary solution obtained in the previous subsection. On the assumption that a vortex sheet is a simple closed curve close to a circle, we rewrite the nonstationary problem (5) and (6) in the complex form that is easier to handle; without a loss of generality, we may assume that the vortex sheet $\Gamma(t)$ is close to unit circle with center O:

$$\Gamma(t)=\left\{(1+r(\lambda,t))e^{i\lambda} \mid -\pi\le\lambda<\pi\right\}$$

where $r(\lambda,t)$ is a real-valued smooth periodic function of the period 2π in λ. Then, the stationary solution is

(37) $$\Gamma_0=\left\{e^{i\lambda} \mid -\pi\le\lambda<\pi\right\},\quad \Omega_0=1$$

And we put

(38) $$U(\lambda,t)=e^{-i\lambda}\{V_1(\lambda,t)+iV_2(\lambda,t)\} =\frac{1}{2\pi i}v.p.\int_{-\pi}^{\pi}\frac{e^{-i\lambda}}{e^{-i\lambda}-e^{-i\lambda'}}\cdot\frac{\Omega(\lambda',t)}{1+p(\lambda,\lambda',t)}d\lambda'$$

where the notation $v.p.\int$ stands for Cauchy's principal value of the integral and

(39) $$p(\lambda,\lambda',t)=\frac{r(\lambda,t)e^{-i\lambda}-r(\lambda',t)e^{-i\lambda'}}{e^{-i\lambda}-e^{-i\lambda'}}$$

Then we can rewrite (5) and (6) by

(40) $$\frac{\partial r}{\partial t}=\mathrm{Re}.U-\frac{\frac{\partial r}{\partial\lambda}}{1+r}\mathrm{Im}.U$$

(41) $$\frac{\partial\Omega}{\partial t}+\frac{\partial}{\partial\lambda}\left(\frac{1}{1+r}\Omega\,\mathrm{Im}.U\right)=0\,.$$

To estimate the singular integral (38), we introduce the Hilbert transform H defined by

(42) $$H[f](\lambda)=\frac{1}{\pi i}v.p.\int_{-\pi}^{\pi}\frac{e^{-i\lambda}}{e^{-i\lambda}-e^{-i\lambda'}}f(\lambda')d\lambda'$$

Then we have

Lemma 3. *Let* $z_n(\lambda)=e^{in\lambda}$. *Then,*

(43) $$H[z_n](\lambda)=-i\,\mathrm{sgn}(n)e^{in\lambda}=\begin{cases}-ie^{in\lambda} & n\ge 0\\ ie^{in\lambda} & n<0\end{cases}$$

In addition, for p *given by* (39)

(44) $$H[p(\lambda,\cdot,t)](\lambda)=i\left(H\left[\frac{\partial r(\cdot,t)}{\partial\lambda}\right](\lambda)-r(\lambda,t)\right).$$

Proof. Since (43) is well known, it may be sufficient to prove (44). Expanding r in Fourier series: $r(\lambda,t)=\sum_n r_n(t)e^{in\lambda}$, we get

(45) $$p=-\sum_{n=1}^{\infty}r_{n+1}\left(\sum_{k=1}^{n}e^{i(n+1-k)\lambda}e^{ik\lambda'}\right)+\sum_{n=0}^{\infty}r_{-n}\left(\sum_{k=0}^{n}e^{-i(n-k)\lambda}e^{-ik\lambda'}\right).$$

Hence it follows from (43) that

(46)
$$H[p(\lambda,\cdot,t)](\lambda) = i\sum_{n=1}^{\infty} n r_{n+1} e^{i(n+1)\lambda} + i\sum_{n=0}^{\infty}(n-1) r_{-n} e^{-in\lambda}.$$
$$= i\sum_{n} |n| r_n e^{in\lambda} - i\sum_{n} r_n e^{in\lambda}$$
$$= iH\left[\frac{\partial r}{\partial \lambda}\right] - ir$$

This completes the proof.

Corollary. *Let* f *be a real-valued function with* $f = \sum_n f_n e^{in\lambda}$ *. Then*

(47)
$$\mathrm{Re}.H[f](\lambda) = -i\sum_{n\neq 0}\mathrm{sgn}(n) f_n e^{in\lambda}\ ,\quad \mathrm{Im}.H[f](\lambda) = -f_0$$

Proof. Noting $\overline{f_n} = f_{-n}$, we immediately obtain (47) from (43).

Putting $\Omega = \Omega_0 + \omega = 1 + \omega$, and expanding U formally with respect to p , we have

(48)
$$U = \frac{1}{2}\left\{H[1] + H[\omega - p] + \sum_{n=1}^{\infty}(-1)^n H\left[(\omega - p)p^n\right]\right\}.$$

Hence from (40) and (41), together with lemma3 and corollary, we can get the linearized equations around (37):

(49)
$$\frac{\partial r}{\partial t} - \frac{1}{2}\frac{\partial r}{\partial \lambda} - \frac{1}{2}\mathrm{Re}.H[\omega] = 0$$

(50)
$$\frac{\partial \omega}{\partial t} - \frac{1}{2}\frac{\partial \omega}{\partial \lambda} + \frac{\partial r}{\partial \lambda} - \frac{1}{2}\frac{\partial}{\partial \lambda}H\left[\frac{\partial r}{\partial \lambda}\right] = 0\ .$$

Differentiating (49) with respect to λ and putting $\rho = \dfrac{\partial r}{\partial \lambda}$, we rewrite (49) and (50) by

(51)
$$\frac{\partial \rho}{\partial t} - \frac{1}{2}\frac{\partial \rho}{\partial \lambda} - \frac{1}{2}\frac{\partial}{\partial \lambda}H[\omega] = 0$$

(52)
$$\frac{\partial \omega}{\partial t} - \frac{1}{2}\frac{\partial \omega}{\partial \lambda} + \rho - \frac{1}{2}\frac{\partial}{\partial \lambda}H[\rho] = 0\ .$$

Then we have

Theorem 2. *If an initial disturbance* $\{\rho^0, \omega^0\}$ *is represented as superposition of* $e^{\pm i\lambda}$ *modes, then the stationary solution* $\{\Gamma_0, \Omega_0\}$ *is stable. If Fourier coefficients of* $\{\rho^0, \omega^0\}$ *contain another modes, then* $\{\Gamma_0, \Omega_0\}$ *is unstable.*

Proof. Fourier coefficients $\{\rho_n, \omega_n\}$ (note $\overline{\rho_n} = \rho_{-n}$, $\overline{\omega_n} = \omega_{-n}$) of (51), (52) satisfy the following simple ordinary differential equations which we integrate easily

(53)
$$\frac{d}{dt}\rho_n(t)-i\frac{n}{2}\rho_n(t)-\frac{|n|}{2}\omega_n(t)=0$$
$$\frac{d}{dt}\omega_n(t)-i\frac{n}{2}\omega_n(t)+\left(1-\frac{|n|}{2}\right)\rho_n(t)=0\ .$$

Then solutions of above equations with initial values $\{\rho_n^0,\omega_n^0\}$ ($n=0,1,2,\cdots$) are as follows.

(54)
$$\rho_0(t)=\rho_0^0\ ,\quad \omega_0(t)=\omega_0^0-\rho_0^0 t\ ,$$

(55)
$$\rho_1(t)=\frac{1}{2}\left\{\left(\rho_1^0-i\omega_1^0\right)e^{it}+\left(\rho_1^0+i\omega_1^0\right)\right\},$$
$$\omega_1(t)=\frac{1}{2}\left\{\left(\omega_1^0+i\rho_1^0\right)e^{it}+\left(\omega_1^0+i\rho_1^0\right)\right\}$$

(56)
$$\rho_2(t)=\left(\rho_2^0+\omega_2^0 t\right)e^{it}\ ,\quad \omega_2(t)=\omega_2^0 e^{it}$$

for $n\geq 3$

(57)
$$\rho_n(t)=\frac{1}{2}\left\{\left(\rho_n^0+\sqrt{\frac{n}{n-2}}\omega_n^0\right)e^{\frac{\sqrt{n(n-2)}}{2}t}+\left(\rho_n^0-\sqrt{\frac{n}{n-2}}\omega_n^0\right)e^{-\frac{\sqrt{n(n-2)}}{2}t}\right\}e^{\frac{ni}{2}t}$$
$$\omega_n(t)=\frac{1}{2}\left\{\left(\omega_n^0+\sqrt{\frac{n-2}{n}}\rho_n^0\right)e^{\frac{\sqrt{n(n-2)}}{2}t}+\left(\omega_n^0-\sqrt{\frac{n-2}{n}}\rho_n^0\right)e^{-\frac{\sqrt{n(n-2)}}{2}t}\right\}e^{\frac{ni}{2}t}\ .$$

Hence we can obtain the required result.

Remark. Contrary to the above results, the stationary flow defined as (8) is unstable for any Fourier mode of initial disturbance.

2. Nonstationary problem

C. Sulem, P.L. Sulem, C. Bardos and U Frisch [5] prove the local existence for the nonlinear problem of the vortex sheet close to a straight line on the basis of an abstract Cauchy-Kowalewski theorem in the formulation of Nishida [3]. Existance theorem for our cases also relies on [3]; our proof seems to be simpler than that in [5]. To estimate singular integral operators, we use a method of Fourier expansion; this idea is suggested by [2].

We introduce a scale of Banach spaces.

Definition. Let B_0 denote the Banach space of 2π -periodic functions: for $f=\sum_n f_n e^{in\lambda}$ satisfying
$$\|f\|_0=\sum_n |f_n|<\infty\ ,$$

and for $q>0$, let B_q denote the subspace of B_0 with norm

$$\|f\|_q = \sum_n |f_n| e^{q|n|} < \infty .$$

The functions which belong to B_q are analytic in the strip:

$\{\lambda + i\mu \in C \mid \lambda \in R/2\pi Z, |\mu| < q\}$.

Moreover, we have

Lemma 4. *For* $q>0$ *and for* $f,g \in B_q$, $f \cdot g \in B_q$, *and* $\frac{\partial f}{\partial \lambda} \in B_{q'}$ $(0 \le q' < q)$ *with*

(58) $$\|f \cdot g\|_q \le \|f\|_q \|g\|_q$$

(59) $$\left\|\frac{\partial f}{\partial \lambda}\right\|_{q'} \le \frac{e^{-1}}{q-q'}\|f\|_q$$

Proof. From $f(\lambda)g(\lambda) = \sum_n \left(\sum_m f_{n-m} g_m\right) e^{in\lambda}$, we get

$$\|f \cdot g\|_q = \sum_n e^{q|n|}\left|\sum_m f_{n-m} g_m\right|$$
$$\le \sum_{m,n} e^{q|n-m|}|f_{m-n}| e^{q|m|}|g_m| = \|f\|_q \|g\|_q$$

Inequality: $\lambda e^{-\varepsilon\lambda} \le (\varepsilon e)^{-1}$ ($\varepsilon > 0$) implies

$$\left\|\frac{\partial f}{\partial \lambda}\right\|_{q'} = \sum_n |n| e^{q'|n|}|f_n| = \sum_n |n| e^{-(q-q')|n|}|f_n| e^{q|n|}$$
$$\le \frac{e^{-1}}{q-q'}\sum_n |f_n| e^{q|n|} = \frac{e^{-1}}{q-q'}\|f\|_q$$

This proves lemma4.

Theorem3. *Let* $q_0 > 0$. *If initial conditions such that the analytic continuations of* r^0, $\frac{\partial r^0}{\partial \lambda}$, Ω^0 *belong to* B_{q_0} *with*

$$\|r^0\|_0 ,\ \left\|\frac{\partial r^0}{\partial \lambda}\right\|_{q_0} < \frac{1}{2},$$

then, there exists a constant K *such that for* $|t| < K(q_0 - q)$ ($0 < q < q_0$), *the system* (40), (41) *has a unique solution* $\{r, \Omega\}$ *which is holomorphic function of* t *with value in* $B_q \times B_q$.

Proof. We apply an abstract Cauchy-Kowalewski theorem to the system:

(60) $$\frac{\partial r}{\partial t} = \mathrm{Re}.U - \frac{\rho}{1+r}\mathrm{Im}.U$$
$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial \lambda}\left(\mathrm{Re}.U - \frac{\rho}{1+r}\mathrm{Im}.U\right)$$
$$\frac{\partial \Omega}{\partial t} + \frac{\partial}{\partial \lambda}\left(\frac{1}{1+r}\Omega\,\mathrm{Im}.U\right) = 0 ,$$

where $\rho = \dfrac{\partial r}{\partial \lambda}$, and U is defined by (38). We write the right-hand side of the above system by $F(v(t), t)$ for a triplet $v = \{r, \rho, \Omega\}$. Then, to check the hypothesis of an abstract Cauchy-Kowalewski theorem [3], it is sufficient to prove the following condition:

Let M and δ be given positive constants. For any $0 \le q' < q < q_0$ and all $v, \tilde{v} \in B_q$ with $\|v\|_q, \|\tilde{v}\|_q < C$ and for any $|t| < \delta$,

(61) $$\|F(v,t) - F(\tilde{v},t)\|_{q'} \le \frac{c}{q - q'}\|v - \tilde{v}\|_q$$

where the constant c depends only on C.

Expanding U: $U = \sum_{n=0}^{\aleph}(-1)^n H[p^n \Omega]$, where p and H defined by (39) and (42), respectively, and noting (59) we can immediately get (61) from the following lemma.

Lemma5. *Let q be a nonnegative constant. For*

$$M = \max\left\{\|r\|_0, \|\tilde{r}_0\|, \left\|\frac{\partial r}{\partial \lambda}\right\|_q, \left\|\frac{\partial \tilde{r}}{\partial \lambda}\right\|_q\right\} < \frac{1}{2}$$

and $\|\Omega\|_q$, $\|\tilde{\Omega}\|_q$ bounded, the following inequalities hold:

(62) $$\left\|H[p^n \Omega]\right\|_q \le (2M)^n \|\Omega\|_q$$

(63) $$\left\|H[p^n \Omega] - H[\tilde{p}^n \tilde{\Omega}]\right\|_q \le c_n\left(\|r - \tilde{r}\|_0 + \left\|\frac{\partial r}{\partial \lambda} - \frac{\partial \tilde{r}}{\partial \lambda}\right\|_q + \|\Omega - \tilde{\Omega}\|_q\right)$$

for $n \ge 1$, where $c_n = (2M)^{n-1}\left(n\|\Omega\|_q + 2M\right)$.

Proof. Using expansion $p(\lambda, \lambda') = \sum_{k,m} p_{k,m} e^{ik\lambda} e^{im\lambda'}$, where

$$p_{k,m} = \begin{cases} -r_{k+m} & k, m \ge 1 \\ r_{k+m} & k, m \le 0 \\ 0 & otherwise \end{cases}$$

and $|r_{-n}| = |\overline{r_n}| = |r_n|$, derived from (45), we get

(64) $$\sum_{k,m} |p_{k,m}| e^{q|k+m|} = |r_0| + 2\sum_{n=1}^{\infty} n|r_m| e^{qn} = |r_0| + \left\|\frac{\partial r}{\partial \lambda}\right\|_q$$

On the other hand, by (43) we have

$$\begin{aligned} & H\left[p(\lambda, \cdot)^n \Omega\right](\lambda) \\ & = \sum_{k_1, \cdots, k_n, m_1, \cdots, m_n, l} -i\,\mathrm{sgn}(k_1 + \cdots + k_n) p_{k_1, m_1} \cdots p_{k_n, m_n} \Omega_l e^{i(k_1 + \cdots + k_n + m_1 + \cdots + m_n + l)\lambda} \end{aligned}$$

Hence we have

$$\left\| H\left[p^n\Omega\right]\right\|_q = \sum_N \left(\sum_{k_1+\cdots+k_n+m_1+\cdots+m_n+l=N} \left| p_{k_1,m_1} \cdots p_{k_n,m_n} \right\| \Omega_l \right| \Bigg) e^{q|N|}$$

$$= \sum_{k_1,\cdots,k_n,m_1,\cdots,m_n,l} \left| p_{k_1,m_1} \right| e^{q|k_1+m_1|} \cdots \left| p_{k_{1n},m_n} \right| e^{q|k_n+m_n|} \left|\Omega_l\right| e^{q|l|}$$

$$\leq (2M)^n \left\|\Omega\right\|_q$$

This shows (62). From this, (64) and

$$p(\lambda,\lambda')^n - \widetilde{p}(\lambda,\lambda')^n = \{p(\lambda,\lambda') - \widetilde{p}(\lambda,\lambda')\} \sum_{j=0}^{n-1} p(\lambda,\lambda')^{n-1-j}\, \widetilde{p}(\lambda,\lambda')^j$$

we obtain

$$\left\| H\left[p^n\Omega\right] - H\left[\widetilde{p}^n\widetilde{\Omega}\right]\right\|_q \leq \left\| H\left[\left(p^n - \widetilde{p}^n\right)\Omega\right]\right\|_q + \left\| H\left[\widetilde{p}^n\left(\Omega - \widetilde{\Omega}\right)\right]\right\|_q$$

$$\leq \sum_{k,m} \left| p_{k,m} - \widetilde{p}_{k,m} \right| e^{q|k+m|} n (2M)^{n-1} \left\|\Omega\right\|_q + (2M)^n \left\|\Omega - \widetilde{\Omega}\right\|$$

$$\leq c_n \left(\left\| r - \widetilde{r} \right\|_0 + \left\| \frac{\partial r}{\partial\lambda} - \frac{\partial\widetilde{r}}{\partial\lambda} \right\|_q + \left\|\Omega - \widetilde{\Omega}\right\|_q \right)$$

This shows (63).

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