

## Honsberger's Algorithm for One-dimensional Tilings with a Three-set

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Let  $p$  and  $q$  be a pair of positive integers. According to an algorithm of Honsberger, a finite interval of the set of integers can be partitioned into congruent copies of a 3-set  $\{0, p, p+q\}$ . Let  $f(p, q)$  be the smallest number of copies of  $\{0, p, p+q\}$  to partition an interval by the algorithm. If  $3p \leq q$  then  $f(p, q)$  is explicitly determined. On the other hand, if  $p < q < 3p$  then the behavior of  $f(p, q)$  is highly complex.

## 1. Introduction

Let  $\mathbf{Z}$  be the set of integers. Let  $T$  be a finite set of  $\mathbf{Z}$ . We call  $T$  an  $n$ -set if the cardinality of  $T$  is  $n$ . We say that  $T'$  is a copy of  $T$  if  $T'$  and  $T$  are congruent with each other. For example, if  $T = \{0, 2, 5\}$  and  $T' = \{7, 10, 12\}$  then  $T'$  is a copy of  $T$ , because  $T'$  is a mirror image of  $T$ . For a set  $X \subseteq \mathbf{Z}$ , we say that  $T$  tiles  $X$  if  $X$  can be partitioned into copies of  $T$ . It is known that every 3-set  $T \subset \mathbf{Z}$  tiles  $\mathbf{Z}$ .<sup>4),6)</sup> In this paper, we focus on a simple but impressive algorithm found by Honsberger.<sup>3)</sup> Actually, a finite interval in  $\mathbf{Z}$  can be partitioned into copies of a given 3-set by the algorithm. For a pair of positive integers  $p$  and  $q$ , let  $T_{p,q}$  denote a 3-set  $\{0, p, p+q\}$ . For any set  $T \subseteq \mathbf{Z}$  and any integer  $x$ , we denote by  $T(x)$  the image of  $T$  under the translation that moves 0 to  $x$ . For example,  $T_{2,3}(7) = \{0+7, 2+7, 5+7\} = \{7, 9, 12\}$ . Note that any copy of  $T_{p,q}$  is denoted by  $T_{p,q}(x)$  or  $T_{q,p}(x)$  with some integer  $x$ .

**Algorithm.** Let  $p \leq q$ .

Step 0. Set  $O = \emptyset$ .  $O$  is regarded as an occupied set.

Step 1. Set  $U = [0, \infty) \setminus O$ . Set  $x = \min U$ . If  $T_{p,q}(x) \subset U$ , then set  $T = T_{p,q}(x)$ , else set  $T = T_{q,p}(x)$ . It turns out that  $T$  is contained in  $U$ . We regard  $T$  as a part of a partition and add  $T$  to  $O$ . If  $O$  is now a finite interval, then stop. Repeat Step 1.

The algorithm has two important properties. Firstly,  $T_{q,p}(x)$  is always unoccupied during the procedure; secondly, a finite interval is partitioned in the end. For example, Figure 1 shows that  $T_{2,3}$  tiles the interval  $[0, 17]$  according to the algorithm.

First, we show that the above algorithm works without contradiction. We claim that  $T_{q,p}(x)$  is always contained in  $U$ . Suppose that  $T_{q,p}(x) \cap O \neq \emptyset$ . Let  $y = x + q$ . Since every element greater than or equal to  $x + p + q$  is not yet occupied, we have  $y \in T_{q,p}(x) \cap O$ . Since  $y - q$  is still unoccupied,  $y - p$  was occupied at the same time when  $y$  was occupied with some copy  $T_{q,p}(z)$ . However, according to the algorithm,  $T_{p,q}(z)$  should be chosen instead of  $T_{q,p}(z)$ , a contradiction.

Secondly, we show that a finite interval is partitioned. In each step, let us define  $\tilde{U}$  as  $\{v \in [0, w) : v + x \in U\}$ , which is a pattern arising at the boundary between  $U$  and  $O$ . Note that  $\tilde{U}$  is one of  $2^{p+q-1}$  possible patterns. Now, for a given pattern  $\tilde{U}$ , we want to specify the most recently added copy  $T$ . Let  $y = \max O$  and  $z = y - (p + q)$ . Since  $T$  contains  $z$  as its smallest element, we may assume  $T = T_{p,q}(z)$  or  $T_{q,p}(z)$ . Before  $T$  is chosen, both  $z$  and  $z + q$  were unoccupied. Hence, if  $z + q$  is still unoccupied then  $T = T_{p,q}(z)$ , and otherwise  $T = T_{q,p}(z)$ . We have shown that each pattern  $\tilde{U}$  has its previous pattern uniquely. It follows that the initial pattern returns in finite steps, as required.

Let  $f(p, q)$  be the number of copies of  $T_{p,q}$  for a partition of a finite interval according to the algorithm. For example, Fig. 1 shows that  $f(2, 3) = 6$ . In this paper, we study  $f(p, q)$ .

Whenever  $x$  is set in the procedure of the algorithm, we have  $[0, x - 1] \cap U = \emptyset$ . We also have  $[x + p + q, \infty) \subseteq U$ , since every previously chosen copy of  $T_{p,q}$  has its leftmost point less than  $x$ . Let us denote a pattern on the interval  $I_x = [x, x + p + q)$  by a sequence  $(a_1, b_1; a_2, b_2; \dots;$

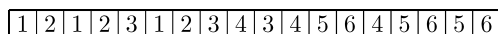


Fig. 1.  $T_{2,3}$  tiles the interval  $[0, 17]$ .

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$a_i, b_i$ ), where the first  $a_1$  consecutive integers starting from  $x$  are unoccupied, and the following  $b_1$  consecutive integers are occupied, and the following  $a_2$  consecutive integers are unoccupied, and so on, and  $b_l$  is the number of the rightmost consecutive integers occupied. We also write a pattern as  $(\dots; (a, b)^r; \dots)$  if a subsequence  $(a, b)$  repeats  $r$  times. Furthermore, we denote the pattern of the initial stage by 0. Let us introduce a pair of transformations  $L$  and  $R$  for all possible patterns on  $I_x$ . If a pattern  $A_1$  results in  $A_2$  by using a translation of  $T_{p,q}$  or  $T_{q,p}$ , we write  $A_1[L] \rightarrow A_2$  or  $A_1[R] \rightarrow A_2$ , respectively. As a matter of fact, according to the algorithm, one of the transformations  $L$  and  $R$  is automatically determined by the current pattern  $A$ .

For example, for  $p=2$  and  $q=3$ , we write  $0[L] \rightarrow (1, 1; 2, 1)$ ,  $(1, 1; 2, 1)[L] \rightarrow (1, 2)$ ,  $(1, 2)[R] \rightarrow (1, 1)$ ,  $(1, 1)[L] \rightarrow (2, 1)$ ,  $(2, 1)[R] \rightarrow (1, 2; 1, 1)$ ,  $(1, 2; 1, 1)[R] \rightarrow 0$ . Furthermore, combining these lines, we simply write  $0[L] \rightarrow (1, 1; 2, 1)[L] \rightarrow (1, 2)[R] \rightarrow (1, 1)[L] \rightarrow (2, 1)[R] \rightarrow (1, 2; 1, 1)[R] \rightarrow 0$ , or  $0[R^2 L R L^2] \rightarrow 0$ .

We will note two simple facts on the algorithm.

**Fact 1.** Let  $1 \leq p \leq q$ . Let  $k$  be the greatest common divisor of  $p$  and  $q$ . Then  $f(p, q) = kf(p/k, q/k)$ .

Indeed, a set  $k\mathbf{Z} + i$  is partitioned into copies of  $T_{p,q}$  as in the same way as  $\mathbf{Z}$  is partitioned into copies of  $T_{p/k, q/k}$  for any  $i$  with  $0 \leq i \leq k-1$ .

Let a pattern  $A$  be  $(a_1, b_1; a_2, b_2; \dots; a_p, b_l)$ . We denote by  ${}^tA$  the transpose  $(b_p, a_p; b_{l-1}, a_{l-1}; \dots; b_1, a_1)$  of  $A$ .

**Fact 2.** If  $A[L(\text{or } R)] \rightarrow B$  then  ${}^tB[R(\text{or } L)] \rightarrow {}^tA$ .

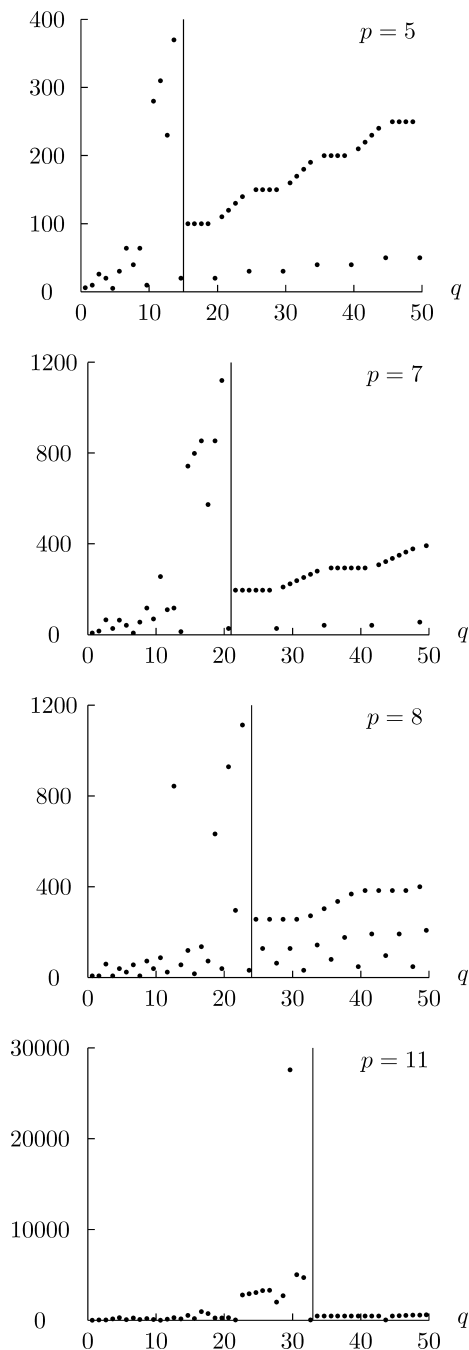
It follows from Fact 2 that the algorithm yields symmetric tilings. In particular, we have  $f(p, q)$  is even if  $p \neq q$ . It is also noted that  $f(p, q) \leq 2^{p+q-2}$  and it is indicated that  $f(p, q)$  may be  $O(pq)$ .<sup>2)</sup>

**Remark 1.** Let  $g(p, q)$  be the smallest number of copies of  $T_{p,q}$  tiling a finite interval  $\subseteq \mathbf{Z}$ . By definition, we have  $g(p, q) \leq f(p, q)$ . In general,  $g(p, q) \neq f(p, q)$ . In fact, it is known that  $g(p, q) = O(p+q)$ ,<sup>1),5)</sup> whereas  $f(p, q) > pq$  for any pair of relatively prime integers  $p$  and  $q$  with  $3p < q$ , as shown in the following sections.

## 2. Periodic behaviors of $f(p, q)$

Typical examples of  $f(p, q)$  calculated by computer are shown in Fig. 2. The following qualitative property can be observed.

**Observation.** If  $p < q < 3p$  then  $f(p, q)$  seems almost



**Fig. 2.**  $f(p, q)$  are dotted for a fixed  $p$  and  $1 \leq q \leq 50$ . The vertical lines  $q=3p$  are shown. These lines seem to be the thresholds between order and disorder.

unpredictable. On the other hand, if  $3p < q$  then  $f(p, q)$  follows a strict regularity.

In fact, an explicit formula for  $f(p, q)$  is obtained within a range of regularity. For  $p=1$ , we have  $f(1, 1)=1, f(1, q)=2 \lceil q/2 \rceil$  for  $q \geq 2$ . By Fact 1 in Section 1, it is sufficient for us to consider the case  $p$  and  $q$  are relatively prime.

**Theorem 1.** *Let  $p$  and  $q$  be a pair of relatively prime integers with  $3p < q$ . Let  $k$  be an integer with  $k \geq 2$  such that  $(2k-1)p < q < (2k+1)p$ . Then*

$$f(p, q) = \begin{cases} 2kp^2 & \text{for } (2k-1)p < q < 2kp \\ 2kp^2 + 2p(q - 2kp) & \text{for } 2kp < q < (2k+1)p. \end{cases}$$

**Proof.** We will prove the theorem by following the algorithm step by step.

Case 1.  $(2k-1)p < q < 2kp$ .

We may assume  $q = (2k-1)p + i$  with  $1 \leq i \leq p-1$ . Let us define  $A_{i,j} = (p-j, 2(k-1)p+i)$  and  $B_{i,j} = (i, j; 2p-j, p; (p, p)^{k-2})$  for  $1 \leq j \leq p-1$ .

**Claim 1.**  $0[R^{(k-1)p+i}L^{kp}] \rightarrow A_{i,j}$ .

We have  $0[L^{kp}] \rightarrow (i, p; (p, p)^{k-1})[R^i] \rightarrow ((p, p)^{k-2}, p, p+i; p-i, i)[R^{(k-1)p}] \rightarrow A_{i,j}$ , as required.

**Claim 2.**  $A_{i,j}[L^{(k-1)p+j}R^{p-j}] \rightarrow B_{i,j}$  for  $1 \leq j \leq p-1$ .

We have  $A_{i,j}[R^{p-j}] \rightarrow (j, p-j)^2[L^j] \rightarrow (2(k-1)p+i, j)[L^{(k-1)p}] \rightarrow B_{i,j}$ , as required.

**Claim 3.**

(i) If  $i+j=p$  then  $B_{i,j}[R^{kp}L^i] \rightarrow 0$ .

(ii) If  $i+j < p$  then  $B_{i,j}[R^{(k-1)p+i+j}L^{p-j}] \rightarrow A_{i,i+j}$ .

(iii) If  $i+j > p$  then  $B_{i,j}[R^{(k-1)p}L^{p-j}R^{i+j-p}] \rightarrow A_{i,i+j-p}$ .

For (i), we have  $B_{i,j}[L^i] \rightarrow ((p, p)^{k-1}, p, i)[R^{kp}] \rightarrow 0$ , as required.

For (ii), we have  $B_{i,j}[L^i] \rightarrow (p-(i+j), i; (p, p)^{k-1}, p, i)[L^{p-(i+j)}] \rightarrow (j, p-(i+j); i, p; (p, p)^{k-2}, p, i; j, p-(i+j))[R^j] \rightarrow (i, p; (p, p)^{k-1}, i, j)[R^i] \rightarrow ((p, p)^{k-2}, p, p+i+j; p-(i+j), i)[R^{(k-1)p}] \rightarrow A_{i,i+j}$ , as required.

For (iii), we have  $B_{i,j}[R^{i+j-p}] \rightarrow (p-j, j; 2p-j, p; (p, p)^{k-3}, p, i+j; 2p-(i+j), i+j-p)[L^{p-j}] \rightarrow ((p, p)^{k-2}, p, i+j; 2p-(i+j), i)[R^{(k-1)p}] \rightarrow A_{i,i+j-p}$ , as required.

Since  $p$  and  $q$  are relatively prime,  $p$  and  $i$  are relatively prime as well. Hence, by Claim 1, 2 and 3, we have a sequence of patterns  $0 \rightarrow A_{i,i} \rightarrow B_{i,i} \rightarrow A_{i,2i} \rightarrow B_{i,2i} \rightarrow \dots \rightarrow A_{i,(p-1)i} \rightarrow B_{i,(p-1)i} \rightarrow 0$ , where the indices are considered modulo  $p$ . By counting the number of transformations in the sequence, we have

$$f(p, q) = \{kp + (k-1)p + i\}$$

$$\begin{aligned} & + \sum_{j=1}^{p-1} \{(p-j) + ((k-1)p+j)\} \\ & + (i+kp) \\ & + \sum_{j=1}^{p-i-1} \{(p-j) + ((k-1)p+i+j)\} \\ & + \sum_{j=p-i+1}^{p-1} \{(i+j-p) + (p-j) + (k-1)p\} \\ & = 2kp^2, \end{aligned}$$

as required.

Case 2.  $2kp < q < (2k+1)p$ .

We may assume  $q = 2kp + i$  with  $1 \leq i \leq p-1$ . Let us define  $C_j = (p-j, j)^3$  and  $D_{i,j} = (p+i+j, p-j; p+j, p-j; (p, p)^{k-2})$  for  $1 \leq j \leq p-1$ .

**Claim 4.**  $0[R^{kp+i}L^{kp}] \rightarrow C_i$ .

We have  $0[L^{kp}] \rightarrow (p+i, p; (p, p)^{k-1})[L^i] \rightarrow (p-i, p+i; (p, p)^{k-1}, p, i)[R^{p-i}] \rightarrow ((p, p)^{k-1}, i, 2p-i)[R^{(k-1)p}] \rightarrow (i, 2p-i; i, 2(k-1)p)[R^i] \rightarrow (i, 2(k-1)p+i; p-i, i)[R^i] \rightarrow C_i$ , as required.

**Claim 5.**  $C_j[L^{kp-2j}] \rightarrow D_{i,j}$  for  $1 \leq j \leq p-1$ .

We have  $C_j[L^{p-j}] \rightarrow (p-j, j; 2(k-1)p+i, p-j)[L^{p-j}] \rightarrow ((2k-3)p+i+j, p-j; p+j, p-j)[L^{(k-2)p}] \rightarrow D_{i,j}$ , as required.

**Claim 6.**

(i) If  $i+j=p$  then  $D_{i,j}[R^{kp+i}L^{2p-i}] \rightarrow 0$ .

(ii) If  $i+j < p$  then  $D_{i,j}[R^{(k-1)p+2i+j}L^jR^{p-(i+j)}L^{i+j}] \rightarrow C_{i+j}$ .

(iii) If  $i+j > p$  then  $D_{i,j}[R^{(k-1)p+i+j}L^{i+j-p}R^{p-j}L^{p-i}R^{i+j-p}L^p] \rightarrow C_{i+j-p}$ .

For (i), we have  $D_{i,j}[L^p] \rightarrow (2p-i, i; (p, p)^{k-1})[L^{p-i}] \rightarrow (i, p; (p, p)^{k-1}, p+i, p-i)[R^i] \rightarrow ((p, p)^{k-1}, p, p+i)[R^{kp}] \rightarrow 0$ , as required.

For (ii), we have  $D_{i,j}[L^{i+j}] \rightarrow (p-(i+j), p+i; p+j, p-j; (p, p)^{k-2}, p, i+j)[R^{p-(i+j)}] \rightarrow (p+j, p-j; (p, p)^{k-2}, i+j, 2p-(i+j))[L^j] \rightarrow (p-j, p; (p, p)^{k-2}, i+j, 2p-(i+j); p+i, j)[R^{p-j}] \rightarrow ((p, p)^{k-2}, i+j, 2p-(i+j); i+j, 2p-j)[R^{(k-2)p}] \rightarrow (i+j, 2p-(i+j); i+j, 2(k-1)p-j)[R^{i+j}] \rightarrow (i+j, 2(k-1)p+i; p-(i+j), i+j)[R^{i+j}] \rightarrow C_{i+j}$ , as required.

For (iii), we have  $D_{i,j}[L^p] \rightarrow (i+j-p, p-j; p+j, p-j; (p, p)^{k-1})[R^{i+j-p}] \rightarrow (p-i, i+j-p; p, p-j; (p, p)^{k-1}, p, i+j-p)[L^{p-i}] \rightarrow (p-j, p-i; i+j-p, p-j; (p, p)^{k-1}, p, i+j-p; p-j, p-i)[R^{p-j}] \rightarrow (i+j-p, p-j; (p, p)^k; i+j-p, p-j)[L^{i+j-p}] \rightarrow (p-i, i+j-p; p-j, p; (p, p)^{k-1}, i+j-p, p-j; p-i, i+j-p)[R^{p-j}] \rightarrow (p-j, p; (p, p)^{k-1}, i+j-p, p;$

$p-j, p-i) [R^{p-j}] \rightarrow ((p, p)^{k-1}; i+j-p, 3p-(i+j); i+j-p, p-j) [R^{(k-1)p}] \rightarrow (i+j-p, 3p-(i+j); i+j-p, (2k-1)p-j) [R^{i+j-p}] \rightarrow (i+j-p, 2(k-1)p+i; 2p-(i+j), i+j-p) [R^{i+j-p}] \rightarrow C_{i+j-p}$ , as required.

By the same way as in Case 1, we have a sequence of patterns  $0 \rightarrow C_i \rightarrow D_{i,i} \rightarrow C_{2i} \rightarrow D_{i,2i} \rightarrow \cdots \rightarrow C_{(p-1)i} \rightarrow D_{i,(p-1)i} \rightarrow 0$ . Hence, we have

$$\begin{aligned} f(p, q) &= 2(kp + i) \\ &+ \sum_{j=1}^{p-1} (kp - 2j) \\ &+ (k+2)p \\ &+ \sum_{j=1}^{p-i-1} \{kp + 2(i+j)\} \\ &+ \sum_{j=p-i+1}^{p-1} \{kp + 2(i+j)\} \\ &= 2kp^2 + 2ip, \end{aligned}$$

as required.  $\square$

### 3. Complex behaviors of $f(p, q)$

If  $p < q < 3p$ , it seems that there is little hope to determine  $f(p, q)$  exactly in general. First, we consider a lower bound of  $f(p, q)$ .

**Conjecture 1.**  $f(p, q) \geq pq$  for any pair of relatively prime integers  $p$  and  $q$ .

If  $p$  or  $q$  is at most 1000, Conjecture 1 is true. On the other hand, there exist infinite pairs of relatively prime integers  $p$  and  $q$  with  $f(p, q) = pq$ , so the inequality of the conjecture is tight if it is true.

**Theorem 2.** Let  $p$  and  $q$  be a pair of relatively prime integers with  $k = q - p \geq 1$ . If one of the following conditions (i) and (ii) is satisfied then  $f(p, q) = pq$ .

- (i)  $k = 1$ .
- (ii)  $k$  is odd with  $k \geq 3$  and  $p \equiv \pm 1 \pmod{k}$ .

**Proof.** We only show an outline of the proof.

For (i), we have  $0[R^p L^1 \cdots R^i L^{p-i+1} \cdots R^1 L^p] \rightarrow 0$ . Hence, we have  $f(p, p+1) = p(p+1)$ , as required.

(ii). Let  $m = (k-1)/2$ .

Case 1.  $p \equiv 1 \pmod{k}$ .

Let us define  $A_i = (p+k-2i-1, 1; (1, 1)^i)$  for  $0 \leq i \leq m$  and  $B_i = (k-2i, 1; (1, 1)^{i-1}, 1, p)$  for  $1 \leq i \leq m$ .

**Claim 1.**  $0[L^1 R^{p-1} L^{k+1} \cdots R^{2k} L^{p-k} R^k L^p] \rightarrow A_0$ .

**Claim 2.**  $A_i [L(R(LR)^i L^{k-2i-1})^{(p-1)/k}] \rightarrow B_{i+1}$  for  $0 \leq i \leq m-1$ .

**Claim 3.**  $B_i [L^{1+i} R^{p-1-i} \cdots L^{p-2k+i} R^{2k-i} L^{p-k+i} R^{k-i}] \rightarrow A_i$  for  $1 \leq i \leq m$ .

With noting that  $B_m = {}^t A_m$ , we have a sequence of patterns  $0 \rightarrow A_0 \rightarrow B_1 \rightarrow A_1 \rightarrow \cdots \rightarrow B_m \rightarrow A_m = {}^t B_m \rightarrow {}^t A_{m-1} \rightarrow \cdots \rightarrow {}^t A_1 \rightarrow {}^t B_1 \rightarrow {}^t A_0 \rightarrow 0$ . It follows that  $f(p, q) = pq$  by some calculation.

Case 2.  $p \equiv -1 \pmod{k}$ .

Let us define  $C_i = ((1, 1)^{i-1}, 1, p+k-2i)$  for  $1 \leq i \leq m$  and  $D_i = (p+1, 1; (1, 1)^{i-1}, 1, k-2i-1)$  for  $1 \leq i \leq m$ .

**Claim 4.**  $0[L^{k-1} R^{p-k+1} L^{2k-1} \cdots R^{2k} L^{p-k} R^k L^p] \rightarrow (p+1, k-1) [R^{k-2} (LR^{k-1})^{(p-k+1)/k} L] \rightarrow C_1$ .

**Claim 5.**  $C_i [L^{k-i-1} R^{p-k+i+1} \cdots L^{p-k-i} R^{k+i} L^{p-i} R^i] \rightarrow D_i$  for  $1 \leq i \leq m$ .

**Claim 6.**  $D_i [R^{k-2i-2} L (RL)^i (R^{k-2i-1} L (RL)^i)^{(p-k+1)/k}] \rightarrow C_{i+1}$  for  $1 \leq i \leq m-1$ .

With noting that  $D_m = {}^t C_m$ , we have a sequence of patterns  $0 \rightarrow C_1 \rightarrow D_1 \rightarrow \cdots \rightarrow D_{m-1} \rightarrow C_m \rightarrow D_m = {}^t C_m \rightarrow {}^t D_{m-1} \rightarrow \cdots \rightarrow {}^t D_1 \rightarrow {}^t C_1 \rightarrow 0$ . It follows that  $f(p, q) = pq$  by some calculation.  $\square$

**Conjecture 2.** Let  $p$  and  $q$  be relatively prime integers with  $p < q$ . Then  $f(p, q) = pq$  if and only if  $p$  and  $q$  satisfy one of the conditions of Theorem 2.

If  $p \leq 1000$ , Conjecture 2 is true for any  $q$  with  $p < q$ .

Lastly, upper bounds of  $f(p, q)$  will be discussed. Let  $M(p)$  be the maximum  $f(p, q)$  over  $p < q < 3p$ . In contrast to that  $f(p, q) < p(p+q)$  holds for  $3p < q$ , it is likely that  $M(p)$  is greater than any polynomial. (See Appendix and Fig. 3.)

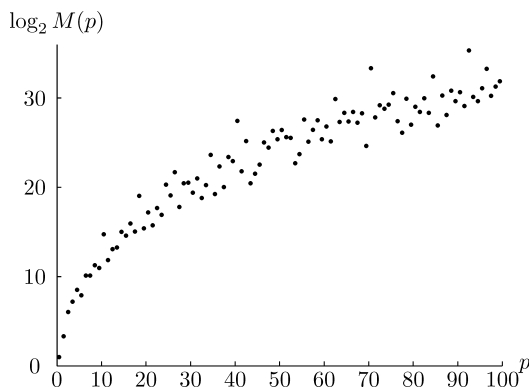
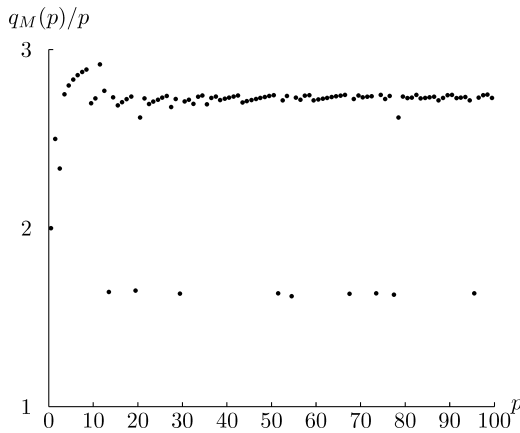


Fig. 3.  $M(p)$  for  $p \leq 100$ .

## Honsberger's Algorithm for One-dimensional Tilings with a Three-set (T. Nakamigawa)

Fig. 4.  $q_M(p)/p$  for  $p \leq 100$ .

**Question 1.** What is an approximate value of  $M(p)$ ?

Let  $q_M(p) = \min\{q: f(p, q) = M(p), p < q < 3p\}$ . (As a matter of fact, the integer  $q$  satisfying  $f(p, q) = M(p)$  and  $p < q < 3p$  is uniquely determined for  $p \leq 100$ .) It appears that  $q_M(p)/p$ 's gather around one or two particular values. (See Appendix and Fig. 4.)

**Question 2.** Does  $q_M(p)/p$  converge as  $p$  tends to infinity?

## References

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Appendix

$M(p)=\max \{f(p, q): p< q< 3p\}$  and  $q_M=\min \{q: f(p, q)=M(p), p< q< 3p\}$  for  $p\leq 100$ .

$p$	$q_M$	$q_M/p$	$M(p)$	$p$	$q_M$	$q_M/p$	$M(p)$
1	2	2.0000	2	51	140	2.7451	90847320
2	5	2.5000	10	52	85	1.6346	52162432
3	7	2.3333	66	53	144	2.7170	48746644
4	11	2.7500	148	54	148	2.7407	6909516
5	14	2.8000	370	55	89	1.6182	14078930
6	17	2.8333	240	56	153	2.7321	205813048
7	20	2.8571	1120	57	155	2.7193	36900660
8	23	2.8750	1112	58	159	2.7414	92861306
9	26	2.8889	2502	59	162	2.7458	190719742
10	27	2.7000	2000	60	163	2.7167	44648220
11	30	2.7273	27610	61	166	2.7213	118255210
12	35	2.9167	3660	62	169	2.7258	37511302
13	36	2.7692	8762	63	172	2.7302	994131054
14	23	1.6429	9854	64	175	2.7344	167510720
15	41	2.7333	33240	65	178	2.7385	345715500
16	43	2.6875	24944	66	181	2.7424	177662958
17	46	2.7059	64464	67	184	2.7463	363737908
18	49	2.7222	34218	68	111	1.6324	160199672
19	52	2.7368	546592	69	188	2.7246	331075386
20	33	1.6500	44160	70	192	2.7429	26437880
21	55	2.6190	149520	71	194	2.7324	1090489538
22	60	2.7273	55220	72	197	2.7361	239850648
23	62	2.6957	211324	73	200	2.7397	615907570
24	65	2.7083	125520	74	121	1.6351	470473958
25	68	2.7200	1298600	75	206	2.7467	644263050
26	71	2.7308	563030	76	207	2.7237	1602135404
27	74	2.7407	3454758	77	211	2.7403	181022996
28	75	2.6786	229180	78	127	1.6282	73553152
29	79	2.7241	1452668	79	207	2.6203	1017990998
30	49	1.6333	1510090	80	219	2.7375	137650000
31	84	2.7097	690308	81	221	2.7284	552611322
32	87	2.7188	2117600	82	224	2.7317	365098768
33	89	2.6970	455136	83	228	2.7470	1076045200
34	93	2.7353	1263814	84	229	2.7262	346676820
35	96	2.7429	13218940	85	232	2.7294	5898401770
36	97	2.6944	625392	86	235	2.7326	129730570
37	101	2.7297	5303432	87	238	2.7356	1324106592
38	104	2.7368	1093184	88	239	2.7159	291985672
39	106	2.7179	10994646	89	243	2.7303	1933884916
40	109	2.7250	8169080	90	247	2.7444	842010570
41	112	2.7317	182549384	91	250	2.7473	1713359830
42	115	2.7381	3665130	92	251	2.7283	581882428
43	118	2.7442	38011570	93	254	2.7312	43973031342
44	119	2.7045	1460360	94	257	2.7340	1189724066
45	122	2.7111	3022200	95	258	2.7158	851119630
46	125	2.7174	6117770	96	157	1.6354	2338882828
47	128	2.7234	34663628	97	265	2.7320	10441598950
48	131	2.7292	23094096	98	269	2.7449	1290405886
49	134	2.7347	84823312	99	272	2.7475	2622144492
50	137	2.7400	44050150	100	273	2.7300	4040857000