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# Semilinear Parabolic Equations with Nonmonotone Nonlinearity

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**ABSTRACT.** A semilinear evolution equation of the type $u_t - \Delta u - g_1(x, t, u) + g_2(x, t, u) = f$ on $\Omega \times (0, T)$ is studied in the space $L^1(\Omega)$, where $\Omega$ is a bounded domain in $R^N$, and $g_1(x, t, r)$ and $g_2(x, t, r)$ are monotone continuous with respect to $r$ and measurable with respect to $x$ and $t$. An existence theorem for the initial value problem associated to this semilinear equation is proved. We then apply this existence result to solve the problem $u_t - \Delta u - u^p + u^q = \nu$ and $u(\cdot, 0) = \mu$ with measures $\nu$ and $\mu$.

## Introduction.

In this paper we study semilinear evolution equations of the type

$$(0.1) \qquad \begin{aligned} & u_t - \Delta u - g_1(x, t, u) + g_2(x, t, u) = f \quad \text{in} \quad Q \,, \\ & u(\cdot, 0) = u_0 \quad \text{in} \quad \Omega \,, \quad u = 0 \quad \text{on} \quad \partial\Omega \times (0, T) \end{aligned}$$

where $Q = \Omega \times (0, T)$ and $\Omega$ is a bounded domain in $R^N$. Here $g_i(x, t, r), i = 1, 2$, are given functions on $Q \times R$ which are measurable in $(x, t)$ and continuous nondecreasing in $r$, and $f$ and $u_0$ are given functions on $Q$ and $\Omega$ respectively. We consider (0.1) in $L^1$ spaces. Namely we shall prove the existence of continuous curve $u$: $[0, T] \to L^1(\Omega)$ satisfying (0.1) in the sense of distributions. We next apply the above existence theorem to the problem

$$(0.2) \qquad \begin{aligned} & u_t - \Delta u - c_1 (u^+)^p + c_2 u |u|^{q-1} = \nu \quad \text{in} \quad Q \\ & u(\cdot, 0) = \mu \quad \text{in} \quad \Omega \,, \quad u = 0 \quad \text{on} \quad \partial\Omega \times (0, T) \,. \end{aligned}$$

Here, $u^+ = \max \{u, 0\}, p, q > 1, c_1, c_2 \geq 0$, and $\mu$ and $\nu$ are given bounded Borel measures on $\Omega$ and $Q$, respectively. If $c_1 = 0$ or $c_2 = 0$, this type of problem has been considered by many authors. Among others, Weissler [17], [18] showed the existence of local solutions of (0.2) in the case where $c_2 = 0$, $\nu = 0$ and $\mu \in L^r(\Omega)$ for $r > N(p-1)/2$, and Baras and Pierre [5] extended some results of [17] to the case where $\mu$, $\nu$ are Borel measures. On the other hand, Baras and Pierre [6] and Brezis and Friedman [7] dealt with (0.2) in the case of $c_1 = 0$. In our argument their results are derived from our result for (0.1) by setting $g_i(x, t, u)$ appropriately. Thus we offer a unified treatment of the type of problem (0.2). Moreover, Baras and Pierre [5] obtained only an "integral" solution which is in some sense the weakest definition of solotions. Our results, however, provide us with more "strict" solutions.

To solve (0.1) we shall employ the standard successive approximation method. In this procedure the estimate of the approximations in an appropriate scale plays an essential role. To obtain that we use a new a priori estimate on integral solutions of (0.1) with

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$g_2\equiv 0$. This type of a priori estimate was first proved by Baras and Cohen [4] for integral solutions to homogeneous equations of the type

$$u_t-\Delta u-g_1(u)=0 \quad \text{in} \quad Q\ .$$
$$u(\cdot,0)=u_0 \quad \text{in} \quad \Omega\ , \quad u=0 \quad \text{on} \quad \partial\Omega\times(0,T]\ .$$

To obtain the a priori estimates on the approximations it is necessary for us to extend their results to the inhomogeneous case.

The outline of this paper is as follows: In Section 1 we present the notations used in this paper and some known results about linear heat equations. In Section 2 we deal with a priori estimates on the integral solutions which are crucial in our arguments. In Section 3 we give the existence theorem of solutions of (0.1) which is our main result. Finally, Section 4 is applications of the existence theorem to the type of problem (0.2).

## 1. Preliminaries.

Throughout this paper $\Omega$ will denote a bounded open set in $R^N(N\geq 1)$ with smooth boundary $\partial\Omega$. Let $T>0$ and $Q=\Omega\times(0,T)$. For $1\leq p<\infty$ $W_p^{2,1}(Q)$ is the Banach space consisting of the elements $u$ of $L^p(Q)$ such that their generalized derivatives $\partial u/\partial x_i$, $\partial^2 u/\partial x_i x_j$ and $\partial u/\partial t$ (written $u_i$, $u_{ij}$ and $u_t$, respectively, in brief) belong to $L^p(Q)$ for $i,j=1,2,\cdots,N$, with the norm

$$|u|_{2,1,p}=|u|_p+|u_t|_p+\sum_{i=1}^{N}|u_i|_p+\sum_{i,j=1}^{N}|u_{ij}|_p$$

where

$$|u|_p=\left(\int_Q |u(x,t)|^p\ dxdt\right)^{1/p}\ .$$

For $1\leq p<\infty$ and $s\geq 0$, $W^{s,p}(\Omega)$ denotes the usual Sobolev space with the norm $\|\cdot\|_{s,p}$ (see [1, Section 7]). We denote the norm of $u$ in $L^p(\Omega)$ by $\|u\|_p$, i.e.

$$\|u\|_p=\left(\int_\Omega |u(x)|^p dx\right)^{1/p}\ .$$

Let $\overset{\circ}{W}{}_p^{2,1}(Q)$ denote the closure of $C_0^\infty(Q)$ in the space $W_p^{2,1}(Q)$ and $W_0^{s,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the space $W^{s,p}(\Omega)$. For convenience of notation we set

$$X=L^\infty(0,T;\ L^1(\Omega))\cap L^1(0,T;\ W_0^{1,1}(\Omega))$$

and

$$X_0=C([0,T];\ L^1(\Omega))\cap L^1(0,T;\ W_0^{1,1}(\Omega))\ .$$

$m_b(\Omega)$ and $m_b(Q)$ will denote the space of bounded signed Radon measures on $\Omega$ and $Q$, respectively. These spaces are equipped with the weak* topology, i.e., $\lim_{n\to\infty}\mu_n=\mu$ in $m_b(\Omega)$ if and only if

$$\lim_{n\to\infty}\int_\Omega \phi d\mu_n=\int_\Omega \phi d\mu$$

for all $\phi \in C_b(\Omega)$ (the space of bounded continuous functions on $\Omega$). Finally, $Y^+$ will denote the nonnegative cone of a vector lattice $Y$.

For reference we collect some well-known results about linear heat equations in the following lemma (For the proofs see e.g. [6, Lemma 3.3] and [7]):

LEMMA 1.1. *For $\mu \in m_b(\Omega)$ and $\nu \in m_b(Q)$ there exists a unique solution $u$ of the problem*

$$\begin{aligned} &u \in X, \quad u_t - \Delta u = \nu \quad \text{in} \quad \mathscr{D}'(Q) \\ &\text{ess}\lim_{t\to+0} u(\cdot, t) = \mu \quad \text{in} \quad m_b(\Omega)\,. \end{aligned} \tag{1.1}$$

*Moreover, if*

$$L:\ m_b(\Omega)\times m_b(Q) \to L^1(Q)$$

*is given by $u = L(\mu, \nu)$ where $u$ is the solution of* (1.1), *then we have*:

(a) *$L$ is an order preserving mapping.*

(b) *For $s, q \geq 1$ with $(2/s)+(N/q) > N+1$ there exists a constant $C = C(s, q, N) > 0$ such that*

$$\|u\|_{L^\infty(0,T;L^1(\Omega))} + \|u\|_{L^s(0,T;W_0^{1,q}(\Omega))} \leq C(|\mu|(\Omega) + |\nu|(Q))\,.$$

(c) *If $1 \leq r < (N+2)/N$ then $L$ is a compact operator from $L^1(\Omega)\times L^1(Q)$ into $L^r(Q)$.*

(d) *If $\nu = f + \nu_1$ with $f \in L^1(Q)^+$ and $\nu_1 \in m_b(Q)$, then we have*

$$u(x, t) = \int_0^t \int_\Omega G(t-s, x, y) f(y, s)\, dy ds + L(\mu, \nu_1)(x, t)$$

*for a.e. $(x, t)$, where $G(t, x, y)$ denotes the Green function of the heat equation with Dirichlet boundary condition.*

The next lemma may be already known, but it seems to me that there is no literature proving it explicitly, so we give the proof of it for completeness.

LEMMA 1.2. *Let $g_2$: $Q\times R \to R$ be a function satisfying the following conditions*: (i) *For each $r \in R$, $g_2(x, t, r)$ is measurable on $Q$*; (ii) *For a.e. $(x, t)$ in $Q$, $g_2(x, t, r)$ is continuous and nondecreasing in $r$ and $g_2(x, t, 0) = 0$; and* (iii) $\sup_{|s|\leq r}|g_2(x, t, s)| \in L^1(Q)$ *for each $r \geq 0$. Then for $u_0 \in L^1(\Omega)$ and $f \in L^1(Q)$ the problem*

$$\begin{aligned} &u \in X_0\,, \quad g_2(\cdot, \cdot, u) \in L^1(Q)\,, \\ &u_t - \Delta u = -g_2(x, t, u) + f \quad \text{in} \quad \mathscr{D}'(Q) \\ &u(\cdot, 0) = u_0 \quad \text{in} \quad \Omega \end{aligned} \tag{1.2}$$

*has a unique solution $u$. Moreover, if we define*

$$S:\ L^1(\Omega)\times L^1(Q) \to L^1(Q)$$

*by $u = S(u_0, f)$ where $u$ is the solution of* (1.2), *and if $\hat{u} = S(\hat{u}_0, \hat{f})$ with $\hat{u}_0 \in L^1(\Omega)$ and $\hat{f} \in L^1(Q)$, then we have*

$$\|(u-\hat{u})^+\|_{L^\infty(0,T;L^1(\Omega))} + |(g_2(\cdot,\cdot,u) - g_2(\cdot,\cdot,\hat{u}))^+|_1 \leq \|(u_0-\hat{u}_0)^+\|_1 + |(f-\hat{f})^+|_1 \tag{1.3}$$

*where $r^+ = \max\{r, 0\}$. In particular, $S$ is an order preserving mapping.*

*Proof*. We shall follow the idea of [9]. For each integer $n$ set $g_{2n}(x,t,r)=\max\{-n, \min\{g_2(x,t,r),n\}\}$. By the Schauder fixed point theorem (see e.g. [16]) there exists $u_n\in X_0$ satisfying

$$(1.4)\qquad \begin{aligned}&(u_n)_t-\Delta u_n+g_{2n}(x,t,u_n)=f \quad \text{in} \quad \mathscr{D}'(Q)\,,\\ &u_n(\cdot,0)=u_0\,.\end{aligned}$$

For $M\geq 0$ and $r\in R$ set

$$p_M(r)=\begin{cases} 1 & \text{if } r>M\,,\\ 0 & \text{if} -M\leq r\leq M\,,\\ -1 & \text{if } r<-M\,.\end{cases}$$

It is well-known that

$$\int_Q (\partial/\partial t-\Delta)u\cdot p_M(u)dxdt\geq -\int_{\Omega_M}|u(x,0)|dx$$

for all $u\in X_0\cap W_1^{2,1}(Q)$, where $\Omega_M=\{x\in\Omega;\ |u(x,0)|>M\}$. Using this inequality we find that

$$\int_{|u_n|>M}|g_{2n}(x,t,u_n)|\,dxdt\leq\int_{|u_n|>M}|f|\,dxdt+\int_{|u_0|>M}|u_0(x)|dx\,.$$

In particular, $|g_{2n}(\cdot,\cdot,u_n)|_1$ is bounded in $n$ and hence it follows from Lemma 1.1 (c) that $\{u_n\}$ is precompact in $L^r(Q)$ for $1\leq r<(N+2)/N$. After extracting a subsequence we may assume that

$$\begin{aligned}&u_n\to u \quad \text{in} \quad L^1(Q)\,, \qquad u_n\to u \quad \text{a.e.,}\\ &g_{2n}(\cdot,\cdot,u_n)\to g_2(\cdot,\cdot,u) \quad \text{a.e..}\end{aligned}$$

On the other hand

$$\begin{aligned}M\,\mathrm{meas}\,[|u_n|>M]&\leq\int_{|u_n|>M}|u_n|\,dxdt\\ &\leq C(|g_{2n}(\cdot,\cdot,u_n)|_1+|f|_1+\|u_0\|_1)\end{aligned}$$

and hence $\sup_n$ meas $[|u_n|>M]\leq$Const.$/M\to 0$ as $M\to\infty$.

Given $\varepsilon>0$ we can therefore choose an $M$ so that

$$\int_{|u_n|>M}|f|\,dxdt+\int_{|u_0|>M}|u_0|dx<\varepsilon/2 \qquad \text{for all } n\geq 1\,.$$

Since $h_M(x,t)\equiv\sup_{|s|\leq M}|g_2(x,t,M)|$ belongs to $L^1(Q)$ by our hypotheses, we can also choose a $\delta=\delta(\varepsilon)>0$ such that

$$\int_A h_M(x,t)dxdt<\varepsilon/2 \quad \text{whenever} \quad A\subset Q \quad \text{and meas } A<\delta\,.$$

Consequently, for such an $A$ we obtain

$$\int_A|g_{2n}(x,t,u_n)|dxdt\leq\int_A h_M(x,t)dxdt+\int_{|u_n|>M}|g_{2n}(x,t,u_n)|dxdt<\varepsilon$$

By the Vitali convergence theorem, $g_{2n}(\cdot,\cdot,u_n)\to g_2(\cdot,\cdot,u)$ in $L^1(Q)$. Therefore, passing to the limit in (1.4) yields that $u$ is the desired solution. The uniqueness will follow from [6, Lemma 3.4].

Finally, to show (1.3) we set $w=u-\hat{u}$. Recalling

$$\int_Q (w_t-\Delta w)\cdot \mathrm{sgn}^+ w\, dxdt \geq \int_\Omega |w(\cdot,T)|dx - \int_\Omega |w(\cdot,0)|dx$$

where $\mathrm{sgn}^+ r=1$ for $r>0$ and $\mathrm{sgn}^+ r=0$ for $r\leq 0$, we can get

$$\|w(\cdot,t)\|_1-\|w(\cdot,0)\|_1 \leq -\int_0^t\int_\Omega [\{g_2(x,t,u)-g_2(x,t,\hat{u})\}^+ + (f-\hat{f})^+]dxdt$$

for all $0\leq t\leq T$, which gives (1.3). $\square$

## 2. A priori estimates on solutions

In this section we will give an a priori estimate on "solutions" of a semilinear parabolic equation with nonmonotone nonlinearity. For this purpose let $g$: $Q\times R^+\to R^+ (R^+=[0,\infty))$ be a function satisfying the following conditions:

(g1) For each $r\in R^+$ $g(x,t,r)$ is measurable on $Q$, and for a.e. $(x,t)$ in $Q$ $g(x,t,r)$ is continuous and nondecreasing in $r$ and $g(x,t,0)=0$.

(g2) For each $r\in R^+$ there exists $\rho_r\in L^{N+1}(Q)$ such that $g(x,t,r)\leq \rho_r(x,t)$ for a.e. $(x,t)$ in $Q$.

(g3) For a.e. $(x,t)$ in $Q$ $g(x,t,r)$ is convex in $r$.

(g4) There exist constants $\gamma>1$ and $a\geq 0$ such that $g(x,t,\lambda r)\geq \lambda^\gamma g(x,t,r)$ for all $\lambda\geq 1$ $r\geq a$ and a.e. $(x,t)\in Q$.

We here note that if $\gamma>1$ then $g(r)=(r^+)^\gamma$ is a typical function satisfying (g1)-(g4).

Now, for $\mu\in m_b(\Omega)^+$, $\nu\in m_b(Q)^+$ and $\lambda\geq 1$ consider the following problem

$$(P_\lambda;\ \mu,\nu) \qquad \begin{cases} (u_\lambda)_t-\Delta u_\lambda-g(x,t,u_\lambda)=\lambda\nu & \text{in } Q\,, \\ u_\lambda(x,0)=\lambda\mu \quad \text{in } \Omega\,, & u_\lambda=0 \quad \text{on } \partial\Omega\times(0,T)\,. \end{cases}$$

Following Baras and Cohen [4] we say that $u_\lambda$ is an integral solution of $(P_\lambda;\ \mu,\nu)$ if $u_\lambda$: $Q\to[0,+\infty]$ is a measurable function satisfying

$$u_\lambda(x,t)=\int_0^t\int_\Omega G(t-s,x,y)g(y,s,u_\lambda(y,s))dyds+V_\lambda(x,t)$$

for a.e. $(x,t)$ in $Q$, where $V_\lambda=L(\lambda\mu,\lambda\nu)$ and $L$ is the operator defined in Lemma 1.1. We say that $U_\lambda$ is a least integral solution of $(P_\lambda;\ \mu,\nu)$ if $U_\lambda$ itself is an integral solution of $(P_\lambda;\ \mu,\nu)$ and whenever $u_\lambda$ is any integral solution of $(P_\lambda;\ \mu,\nu)$ we have $U_\lambda\leq u_\lambda$ a.e. on $Q$. We here remark that $U_\lambda$ may equal infinity identically, so we set

$$T_\lambda^*(\mu,\nu)=\sup\{t\geq 0;\ U_\lambda \text{ is finite for a.e. on } \Omega\times(0,t)\}\,.$$

LEMMA 2.1. *Let $g$, $\hat{g}$: $Q\times R^+\to R^+$ satisfy* (g1). *Then we have*:

(a) *There exists a least integral solution of $(P_\lambda;\ \mu,\nu)$.*

(b) *If $g\leq\hat{g}$ a.e. on $Q\times R^+$ and $U_\lambda$, $\hat{U}_\lambda$ are the corresponding least integral solutions of $(P_\lambda;\ \mu,\nu)$, then $U_\lambda\leq\hat{U}_\lambda$ a.e. on $Q$.*

*Proof.* Let $v_\lambda$ be an arbitrary integral solution of $(P_\lambda;\ \mu,\nu)$ and $\{u^n\}$ be the sequence defined by

$$u_\lambda^n \in X\,,\qquad (u_\lambda^n)_t - \Delta u_\lambda^n = g_n(x,t,u_\lambda^{n-1}) + \lambda\nu \quad \text{in} \quad \mathscr{D}'(Q)\,,$$
$$\text{ess}\lim_{t\to+0} u_\lambda^n(\cdot,0) = \lambda\mu \quad \text{in} \quad m_b(\Omega)\,,$$

and $u_\lambda^0 \equiv 0$ on $Q$, where $g_n = \min\{g, n\}$. By Lemma 1.1 this sequence $\{u_\lambda^n\}$ exists and satisfies

$$u_\lambda^n(x,t) = \int_0^t \int_\Omega G(t-s,x,y) g_n(y,s,u_\lambda^{n-1}) dyds + V_\lambda(x,t)\ .$$

By recurrence we see that $u_\lambda^{n-1} \leq u_\lambda^n \leq v_\lambda$ and $u_\lambda^n \leq \hat{U}_\lambda$ a.e. on $Q$. Set $U_\lambda = \lim_{n\to\infty} u_\lambda^n$ on $Q$. It follows from the monotone convergence theorem that

$$U_\lambda(x,t) = \int_0^t \int_\Omega G(t-s,x,y) g(y,s,U_\lambda) dyds + V_\lambda(x,t)\ .$$

Hence $U_\lambda$ is an integral solution of $(P_\lambda;\ \mu,\nu)$ satisfying $U_\lambda \leq v_\lambda$ and $U_\lambda \leq \hat{U}_\lambda$. □

Now, we give a priori estimates on the least integral solutions of $(P_\lambda;\ \mu,\nu)$ with $\lambda=1$ which is crucial in our arguments.

LEMMA 2.2. *Let* (g1)-(g4) *be satisfied and let* $\mu \in m_b(\Omega)^+$ *and* $\nu \in m_b(Q)^+$. *Assume that* $T^* \equiv T^*_{\lambda_0}(\mu,\nu) > 0$ *for some* $\lambda_0 > 1$. *Then we have*

$$U_1(x,t) \leq \lambda_0^\gamma/(\lambda_0^{\gamma-1}-1)^{\gamma/(\gamma-1)}(V_1(x,t)+a) \qquad \textit{for a.e. } (x,t)\in\Omega\times(0,T^*)\ . \tag{2.1}$$

*Proof.* We shall modify the arguments of [4]. Let $\{\mu_j\} \subset C_0^\infty(\Omega)^+$ and $\{\nu_j\} \subset C_0^\infty(Q)^+$ be sequences such that

$$\mu_j \to \mu \quad \text{in} \quad m_b(\Omega)\,,\qquad \nu_j \to \nu \quad \text{in} \quad m_b(Q)\,,$$
$$\sup_j \|\mu_j\|_1 < +\infty \quad \text{and} \quad \sup_j |\nu_j|_1 < +\infty\ .$$

For $j \in N$ and $\lambda \in [1, \lambda_0]$, let $u_\lambda^{n,j}$ (written $u_\lambda^n$ for simplicity if there is no need for distinction or possibility of confusion) be the sequence given by

$$\begin{aligned} &u_\lambda^n \in W_{N+1}^{2,1}(Q^*)\,,\qquad Q^* \equiv \Omega\times[0,T^*)\,,\\ &(u_\lambda^n)_t - \Delta u_\lambda^n = g(x,t,u_\lambda^{n-1}) + \lambda\nu_j \quad \text{in} \quad Q^*\,,\\ &u^n(\cdot,0) = \lambda\mu_j \quad \text{in} \quad \Omega\,,\qquad u^n = 0 \quad \text{on} \quad \partial\Omega\times[0,T^*)\,, \end{aligned} \tag{2.2}$$

and $u_\lambda^0 \equiv 0$ on $Q^*$. We show that this sequence exists. Indeed, since $g(x,t,u_\lambda^0)=0$ and $\nu_j \in L^{N+1}(Q^*)$, there exists $u_\lambda^1$ satisfying (2.2) with $n=1$ (cf. [11, Theorem 9.1]). By the embedding theorem (cf. [11, Lemma 3.3]) $u_\lambda^1 \in W_{N+1}^{2,1}(Q^*) \subset C(\bar{Q}^*)$ and hence $g(x,t,u_\lambda^1) \leq \rho_r(x,t)$ by (g2) where $r = \sup_{Q^*}|u_\lambda^1(x,t)|$. Thus $g(x,t,u_\lambda^1)$ belongs to $L^{N+1}(Q^*)$, and so by [11, Theorem 9.1] again there exists $u_\lambda^2$ satisfying (2.2) with $n=2$. Inductively, we can obtain the sequence $\{n_\lambda^n\}$ satisfying (2.2) for all $n$.

By recurrence and (g1), (g3) we have

$$\begin{aligned} &0 \leq u_\lambda^n \leq u_\lambda^{n+1} \leq U_\lambda \quad \text{on} \quad Q^*\,,\\ &\lambda u_1^n \leq u_\lambda^n \quad \text{on} \quad Q^*\ . \end{aligned} \tag{2.3}$$

For simplicity write that $\eta_0=\lambda_0^{r-1}/(\lambda_0^{r-1}-1)$. Fix $m\in N$ for the moment. For $\eta\geq\eta_0$ set

$$E_\eta^m=\{(x,t)\in Q^*;\ u_1^m(x,t)>\eta(\tilde{V}_j(x,t)+a)\}$$

where $\tilde{V}_j=L(\mu_j,\nu_j)$, which belongs to $C^{2,1}(Q^*)\cap C(\bar{\Omega}\times[0,T^*])$.

Suppose now that there exists an $\eta_m>\eta_0$ such that $E_{\eta_m}^m\neq\emptyset$. For $\eta\in[\eta_0,\eta_m]$ and $n\geq m$ define

$$g_n^m(\eta)=\inf_{(x,t)\in E_\eta^m}\frac{u_{\lambda_0}^n(x,t)}{u_1^m(x,t)}$$

and

$$w(x,t)=u_{\lambda_0}^{n+1}(x,t)-g_n^m(\eta)^r u_1^m(x,t)+\eta(g_n^m(\eta)^r-g_n^m(\eta))(\tilde{V}_j(x,t)+a)\ .$$

We deduce from (2.3) that

$$1<\lambda_0\leq g_n^m(\eta)\leq\inf_{(x,t)\in E_\eta^m}U_{\lambda_0}(x,t)/u_1^m(x,t)<+\infty \tag{2.4}$$

and from (g4) that

$$g(x,t,u_{\lambda_0}^n)\geq g(x,t,g_n^m(\eta)u_1^m)\geq g_n^m(\eta)^r g(x,t,u_1^m)\quad\text{on}\quad E_\eta^m\ .$$

Hence

$$\begin{aligned}w_t-\Delta w&=g(x,t,u_{\lambda_0}^n)+\lambda_0\nu_j-g_n^m(\eta)^r(g(x,t,u_1^{m-1})+\nu_j)+\eta(g_n^m(\eta)^r-g_n^m(\eta))\nu_j\\&\geq\{\lambda_0-g_n^m(\eta)^r+\eta(g_n^m(\eta)^r-g_n^m(\eta))\}\nu_j\quad\text{on}\quad E_\eta^m\ .\end{aligned}$$

However, by observing that $\lambda_0-s^r+\eta(s^r-s)\geq 0$ whenever $s\geq\lambda_0$ we obtain

$$w_t-\Delta w\geq 0\quad\text{on}\quad E_\eta^m\ .$$

On the other hand we have

$$\begin{aligned}w&\geq g_{n+1}^m(\eta)u_1^m-g_n^m(\eta)^r u_1^m+\eta(g_n^m(\eta)^r-g_n^m(\eta))(\tilde{V}_j+a)\\&\geq(g_n^m(\eta)^r-g_n^m(\eta))\{-u_1^m+\eta(\tilde{V}_j+a)\}\quad\text{on}\quad E_\eta^m\ .\end{aligned}$$

Since $-u_1^m+\eta(\tilde{V}_j+a)\geq 0$ on the parabolic boundary $\partial_p Q^*\equiv(\partial\Omega\times(0,T^*))\cup(\Omega\times\{0\})$, it follows from the above inequality and the definition of $E_\eta^m$ that

$$w\geq 0\quad\text{on}\quad\partial_p E_\eta^m\equiv\partial E_\eta^m/(\Omega\times\{T^*\})\ .$$

Moreover, $w$ belongs to $W_{N+1}^{2,1}(Q^*)$, and hence by the maximum principle (cf. [15], [12]), we have $w\geq 0$ on $E_\eta^m$ for $\eta\in[\eta_0,\eta_m]$.

For $\eta_0\leq\eta\leq\eta'\leq\eta_m$ we have

$$\tilde{V}_j+a\leq(\eta')^{-1}u_1^m\quad\text{on}\quad E_{\eta'}^m$$

and by the fact that $w\geq 0$ on $E_{\eta'}^m$,

$$u_{\lambda_0}^{n+1}\geq g_n^m(\eta)^r u_1^m-(\eta/\eta')(g_n^m(\eta)^r-g_n^m(\eta))u_1^m\quad\text{on}\quad E_{\eta'}^m\ ,$$

which gives

$$g^m_{n+1}(\eta')\geq g^m_n(\eta)^\gamma-(\eta/\eta')(g^m_n(\eta)^\gamma-g^m_n(\eta)) .$$

The sequence $\{g^m_n(\eta)\}^\infty_{n=1}$ is nondecreasing and bounded by (2.3) and (2.4). Its limit $g^m(\eta)=\lim_{n\to\infty}g^m_n(\eta)$ satisfies

$$g^m(\eta')\geq g^m(\eta)^\gamma-(\eta/\eta')(g^m(\eta)^\gamma-g^m(\eta)) .$$

Hence

$$(d/d\eta)g^m(\eta)/(g^m(\eta)^\gamma-g^m(\eta))\geq 1/\eta \quad \text{a.e.} \quad \eta\in[\eta_0, \eta_m] .$$

Integrating on $[\eta_0, \eta_m]$ yields

$$\log(\eta_m/\eta_0)\leq\int_\alpha^\beta(s^\gamma-s)^{-1}ds\leq\int_{\lambda_0}^\beta(s^\gamma-s)^{-1}ds$$

where $\alpha=g^m(\eta_0)$ and $\beta=g_m(\eta_m)$, from which we obtain

$$\eta_m\leq\eta_0\lambda_0/(\lambda_0^{\gamma-1}-1)^{1/(\gamma-1)}=\lambda_0^\gamma/(\lambda_0^{\gamma-1}-1)^{\gamma/(\gamma-1)} .$$

This means that $E^m_\eta=\phi$ whenever $\eta>\lambda_0^\gamma(\lambda_0^{\gamma-1}-1)^{\gamma/(\gamma-1)}$ (written $\tilde\lambda_0$ for simplicity). Consequently, we have

$$u^{m,j}_1\leq\eta(\tilde V_j+a) \quad \text{on } Q^* \quad \text{for all } \eta>\tilde\lambda_0 \quad \text{and} \quad m, j\in N . \tag{2.5}$$

Now set

$$g_k(x, t, r)=\min\{g(x, t, r), k\} , \qquad k\in N .$$

For $k, j\in N$ and $\lambda\geq 1$ let $v^{n,j,k}_\lambda$ be the sequence given by (2.2) with $g_k$ instead of $g$. By recurrence we see that

$$\begin{aligned} &0\leq v^{n,j,k}_\lambda\leq u^{n,j}_\lambda && \text{on } Q^* \\ &v^{n,j,k}_\lambda\leq v^{n+1,j,k}_\lambda\leq v^{n+1,j,k+1}_\lambda && \text{on } Q^* . \end{aligned} \tag{2.6}$$

Hence the limit $v^{j,k}_1(x, t)=\lim_{m\to\infty}v^{m,j,k}_1(x, t)$ exists monotonously for $(x, t)$ in $Q^*$ and we have from Lemma 1.1 (b) that

$$\sup_m\int_{Q^*}v^{m,j,k}_1dxdt\leq C\{|g_k(x, t, v^{m-1,j,k}_1)+\nu_j|_1+\|\mu_j\|_1\}\leq C\{k\, m(Q^*)+|\nu_j|_1+\|\mu_j\|_1\} .$$

Here $m(Q^*)$ denotes the Lebesgue measure of $Q^*$. If follows from Beppo-Levi's theorem that $v^{m,j,k}_1\to v^{j,k}_1$ and $g_k(\cdot, \cdot, v^{m-1,j,k}_1)\to g_k(\cdot, \cdot, v^{j,k}_1)$ in $L^1(Q^*)$ as $m\to\infty$. Passing to the limit in (2.2) with $g_k$ instead of $g$, we see that the limit $v^{j,k}_1$ satisfies

$$\begin{aligned} &v^{j,k}_1\in W^{2,1}_{N+1}(Q^*)\cap L^1(0, T^*; W^{1,1}_0(\Omega)) , \\ &(v^{j,k}_1)_t-\Delta v^{j,k}_1=g_k(x, t, v^{j,k}_1)+\nu_j \quad \text{in } \mathscr{D}'(Q^*) , \\ &v^{j,k}_1(\cdot, 0)=\mu_j \quad \text{in } \Omega . \end{aligned} \tag{2.7}$$

Since $\{g_k(\cdot, \cdot, v^{j,k}_1)\}^\infty_{j=1}$ and $\{\nu_j\}^\infty_{j=1}$ are bounded in $L^1(Q^*)$ and $\{\mu_j\}^\infty_{j=1}$ is bounded in $L^1(\Omega)$, it follows from Lemma 1.1 (c) that $\{v^{j,k}_1\}^\infty_{j=1}$ and $\{\tilde V_j\}^\infty_{j=1}$ are precompact in $L^1(Q^*)$, so we may assume that there exists $v^k_1\in L^1(Q^*)$ such that

$$v^{j,k}_1\to v^k_1 \quad \text{and} \quad \tilde V_j\to V_1 \quad \text{in } L^1(Q^*) \quad \text{and a.e. on } Q^*$$

as $j\to\infty$, where $V_1=L(\mu,\nu)$. Letting $j\to\infty$ in (2.7) and then using Lemma 1.1 (d) yield

$$v_1^k(x,t)=\int_0^t\int_\Omega G(t-s,x,y)g_k(y,s,v_1^k(y,s))dyds+V_1(x,t)$$

for a.e. $(x,t)$ in $Q^*$. However, $v_1^k\leq v_1^{k+1}\leq U_1<+\infty$ on $Q^*$ by (2.3) and (2.6) and hence by the monotone convergence theorem the limit $v_1=\lim_{k\to\infty}v_1^k$ satisfies

$$v_1(x,t)=\int_0^t\int_\Omega G(t-s,x,y)g(y,s,v_1(y,s))dyds+V_1(x,t)$$

for a.e. $(x,t)$ in $Q^*$. By definition, $v_1$ is an integral solution of $(P_1;\ \mu,\nu)$ satisfying $v_1\leq U_1$. Since $U_1$ was the least integral solution of $(P_1;\ \mu,\nu)$, we must have $v_1=U_1$. Consequently, it follows from (2.5) and (2.6) that $U_1=v_1\leq\tilde{\lambda}_0(V_1+a)$ a.e. on $Q^*$. $\square$

Next we give a sufficient condition which ensures that $T_\lambda^*(\mu,\nu)>0$ for some $\lambda>1$. To this end we further assume that the following condition holds:

(g5) There exists a constant $b>0$ such that $g(x,t,b)^{-1/(\gamma-1)}\in L^1_{\text{loc}}(Q)$, where $\gamma$ is the constant appearing in Condition (g4).

Let $g^*$ be the conjugate function of $g$, i.e.

$$g^*(x,t,r)=\sup_{\alpha\geq0}\{\alpha r-g(x,t,\alpha)\}$$

for a.e. $(x,t)$ in $Q$ and $r\geq0$. Following [5] we set

$$Z=\{\theta\in L^\infty(Q)^+;\ \text{supp}\,\theta\text{ is compact and }g^*(x,t,\theta/\xi)\xi\in L^1(Q)\}$$

where

$$\xi(x,t)=\hat{\xi}(x,T-t)\quad\text{for}\quad(x,t)\in Q\quad\text{and}\quad\hat{\xi}=L(0,\theta)\,. \tag{2.8}$$

For $\mu\in m_b(\Omega)^+$ and $\nu\in m_b(Q)^+$ we define

$$N_{g,T}(\mu,\nu)=\sup_{\theta\in Z}\frac{\int_\Omega\xi(\cdot,0)d\mu+\int_Q\xi d\nu}{\int_Q g^*(x,t,\theta/\xi)\xi dxdt}$$

LEMMA 2.3. *Let* (g1)–(g5) *be satisfied. Let* $\mu\in m_b(\Omega)^+$, $\nu\in m_b(Q)^+$, $T>0$ *and* $\lambda\geq1$. *If* $N_{g,T}(\lambda\mu,\lambda\nu)\leq1$, *then* $(P_\lambda;\ \mu,\nu)$ *has an integral solution such that* $T_\lambda^*(\mu,\nu)\geq T$.

*Proof.* Let $V_\lambda=L(\lambda\mu,\lambda\nu)$ as before. We can easily see that

$$\int_Q V_\lambda\theta dxdt=\lambda\int_\Omega\xi(\cdot,0)d\mu+\lambda\int_Q\xi d\nu\leq\int_Q g^*(x,t,\theta/\xi)\xi dxdt$$

for all $\theta\in Z$. Here we used the assumption that $N_{g,T}(\lambda\mu,\lambda\nu)\leq1$. By virtue of [5, Theorem 2.1] $(P_\lambda;\ \mu,\nu)$ has an integral solution $u_\lambda$ such that $u_\lambda\theta\in L^1(Q)$ for all $\theta\in\hat{Z}\equiv\{\theta\in Z;\ g^*(\cdot,\cdot,\alpha\theta/\xi)\xi\in L^1(Q)$ for some $\alpha>1\}$. To show that $T_\lambda^*(\mu,\nu)\geq T$, let $K$ be a compact subset of $\Omega$ and $T_1\in(0,T)$. Set $\xi_1(x,t)=\{(T_1-t)^+\}^{\gamma'}\eta_0(x)^{2\gamma'}$ where $\gamma'=\gamma/(\gamma-1)$ and $\eta_0\in C_0^\infty(\Omega)^+$ satisfying $\eta_0=1$ on $K$. Then, we want to show that $\theta\equiv(-(\xi_1)_t-\Delta\xi_1)^+$ belongs to $\hat{Z}$. Indeed, define $\xi$ by (2.8) where $\theta$ is the function above. The maximum principle (cf. [8]) implies that

$\xi \geq \xi_1$ on $Q$ and easy calculations imply that $\theta \leq C\{(T_1-t)^+\}^{\gamma'-1}\eta_0^{2\gamma'-2}$, and so $\theta^{\gamma'}\xi_1^{1-\gamma'} \leq C$. Here and in what follows $C$ denotes various constants, which need not be the same throughout. On the other hand, (g2)-(g4) deduce that

$$g^*(x, t, r) \leq C\{r + g(x, t, b)^{-1/(\gamma-1)} r^{\gamma'}\},$$

which gives that for all $\alpha > 1$

$$\int_Q g^*(x, t, \alpha\theta/\xi)\xi\, dxdt \leq C \int_Q \{\theta + g(x, t, b)^{-1/(\gamma-1)} \theta^{\gamma'} \xi_1^{1-\gamma'}\} dxdt < +\infty$$

by (g5). Thus we obtain $\theta \in \hat{Z}$. Therefore, $u_\lambda \theta \in L^1(Q)$ implies that $u_\lambda \in L^1(K \times (0, T_1))$. Since $K \subset \Omega$ and $T_1 \in (0, T)$ can be taken arbitrarily, it follows that $u_\lambda \in L^1_{loc}(Q)$, and so $T_\lambda^*(\mu, \nu) \geq T$. $\square$

## 3. Semilinear equations in $L^1$.

In this section we will be concerned with the following problem

$$\text{(3.1)} \qquad \begin{aligned} &u \in X_0, \quad g_i(\cdot, \cdot, u) \in L^1(Q), \quad i = 1, 2, \\ &u_t - \Delta u - g_1(x, t, u) + g_2(x, t, u) = f \quad \text{in } \mathscr{D}'(Q), \\ &u(\cdot, 0) = u_0 \quad \text{in } \Omega, \end{aligned}$$

where $g_i$: $Q \times R \to R$, $f$: $Q \to R$ and $u_0$: $\Omega \to R$ are given functions and $u$ is unknown. We will solve (3.1) in $L^1$ spaces under the following conditions:

(H1) For $r \in R$ and $i=1, 2$, $g_i(x, t, r)$ is measurable on $Q$.

(H2) For a.e. $(x, t)$ in $Q$ and $i=1, 2$, $g_i(x, t, r)$ is continuous and nondecreasing with respect to $r$, $g_2(x, t, 0)=0$ and $g_1(x, t, r) \geq 0$ for all $r \in R$.

(H3) There exist $\gamma > 1$, $\phi \in m_b(\Omega)^+$ and $\psi \in m_b(Q)^+$ which satisfy the following conditions:

(i) $$g_1(x, t, r - w(x, t)) \leq g(r) \equiv r^\gamma$$

for $r \geq 0$ and a.e. $(x, t) \in Q$, where $w = L(\phi, \psi)$ ($L$ is the operator defined in Lemma 1.1).

(ii) $$N_{g,T}(u_0^+ + \phi, f^+ + \psi) = \rho < 1.$$

(iii) $$\int_Q g_1(x, t, \tilde{\rho} h(x, t)) dxdt < +\infty,$$

where $h = L(u_0^+ + \phi, f^+ + \psi)$ and $\tilde{\rho} = (1 - \rho^{\gamma-1})^{-\gamma/(\gamma-1)}$.

(H4) $$\sup_{|r| \leq s} |g_2(x, t, r)| \in L^1(Q) \qquad \text{for } s \geq 0.$$

(H5) $$u_0 \in L^1(\Omega) \quad \text{and} \quad f \in L^1(Q).$$

Our result of this section is the following.

THEOREM 3.1. *Let* (H1)-(H5) *be satisfied. Then,* (3.1) *has a solution.*

*Remarks.* (a) As will be seen in the proof of the theorem, we can replace $g(r) = r^\gamma$ in Condition (H3) with the function $g(x, t, r)$ defined on $Q \times R^+$ satisfying Conditions (g1)-

(g5) in the previous section. In this case, however, (H3)-(iii) must be replaced by

(iii)' $$\int_Q g_1(x, t, \tilde{\rho}(h(x, t)+a))dxdt<+\infty ,$$

where $a$ is the constant appearing in (g4). (b) The requirement for $g_i(x, t, r)$ not to be continuous in $(x, t)$ but to be measurable in $(x, t)$ is essential in applications to equations involving measures (see the next section).

We will prove the theorem under the general conditions stated in the remarks above. We begin with the following

LEMMA 3.2. *Let* (H1)-(H5) *be satisfied. Then the problem*

(3.2) $$\begin{aligned} &z_t-\Delta z-g_1(x, t, z)=f^+ \quad in\ Q ,\\ &z(\cdot, 0)=u_0^+ \quad in\ \Omega , \qquad z=0 \quad on\ \partial\Omega\times(0, T) . \end{aligned}$$

*has the least integral solution z. Moreover, we have*

$$z(x, t)\leq\tilde{\rho}(h(x, t)+a) \qquad for\ a.e.\ on\ Q .$$

*Proof.* Set $\lambda_0=1/\rho>1$. Since

$$N_{g,T}(\lambda_0(u_0^++\phi), \lambda_0(f^++\psi))=\lambda_0\rho=1 ,$$

it follows from Lemma 2.3 that $T^*_{\lambda_0}(u_0^++\phi, f^++\psi)\geq T>0$. Hence, by Lemmas 2.1 and 2.2 there exists a least integral solution $V$ of $(P_1;\ u_0^++\phi, f^++\psi)$ such that

$$V(x, t)\leq\tilde{\rho}(h(x, t)+a) \qquad \text{a.e. on } Q .$$

Set $\tilde{g}_1(x, t, r)=g_1(x, t, r-w(x, t))$ where $w=L(\phi, \psi)$. (H1), (H2) and (H3)-(i) imply that $\tilde{g}_1\leq g$ and $\tilde{g}_1$ satisfies (g1). By Lemma 2.1 there exists a least integral solution $\tilde{U}$ of $(P_1;\ u_0^++\phi, f^++\psi)$ with $g$ replaced by $\tilde{g}_1$ such that $\tilde{U}\leq V$.

On the other hand, it is easy to see that $z=\tilde{U}-w$ is the least integral solution of $(P_1;\ u_0^+, f^+)$ with $g$ replaced by $g_1$. Consequently we have

$$z\leq\tilde{U}\leq V\leq\tilde{\rho}(h+a) \qquad \text{a.e. on } Q . \qquad \square$$

*Proof of Theorem* 3.1. Let $\{u^n\}$ be the sequence given by

(3.3) $$\begin{aligned} &u^n\in X_0 , \qquad g_2(\cdot, \cdot, u^n)\in L^1(Q) ,\\ &(u^n)_t-\Delta u^n=g_1(x, t, u^{n-1})-g_2(x, t, u^n)+f \qquad \text{in } \mathscr{D}'(Q) .\\ &u^n(\cdot, 0)=u_0 \qquad \text{in } \Omega , \end{aligned}$$

and $u^0=-w$ on $Q$. Let $\{v^n\}$ be the sequence given by $(3.3)^+$ which means (3.3) with $u^n, f$ and $u^0$ replaced by $v^n, f^+$ and $u_0^+$ respectively, and $v^0=-w$ on $Q$. First, we shall show that there exist sequences $\{u^n\}$ and $\{v^n\}$ satisfying (3.3) and $(3.3)^+$, respectively, and the following property holds:

(3.4) $$u^n\leq v^n\leq z \qquad \text{a.e. on } Q ,$$

where $z$ is the least integral solution of (3.2). Since $g_1(x, t, -w(x, t))=0$ by (H2) and (H3)-(i), $u^1$ and $v^1$ exist by Lemma 1.2. Moreover, if $S$ is the operator defined in Lemma 1.2, then $u^1=S(u_0,f)\leq S(u_0^+,f^+)=v^1$ and $0=S(0,0)\leq v^1=L(u_0^+,-g_2(\cdot,\cdot,v^1)+f^+)\leq L(u_0^+,f^+)\leq z$. Inductively, assume that (3.3), (3.3)$^+$ and (3.4) hold up to $n-1$. Note that $0\leq g_1(x,t,u^{n-1})\leq g_1(x,t,v^{n-1})\leq g_1(x,t,z)\leq g_1(x,t,\tilde{\rho}(h+a))$ by Lemma 3.2. Hence, Lemma 1.2 together with (H3)-(iii) assures that $u^n$ and $v^n$ exist. Since the operator $S$ has the ordre preserving property, $u^n=S(u_0,g_1(\cdot,\cdot,u^{n-1})+f)\leq S(u_0^+,g_1(\cdot,\cdot,v^{n-1})+f^+)=v^n$ and $v^n\geq 0$. Moreover, it follows from Lemma 1.1 (d) that

$$v^n(x,t)=\int_0^t\int_\Omega G(t-s,x,y)g_1(y,s,v^{n-1})dyds+L(u_0,-g_2(\cdot,\cdot,v^n)+f)(x,t)$$
$$\leq\int_0^t\int_\Omega G(t-s,x,y)g_1(y,s,z)dyds+L(u_0^+,f^+)(x,t)=z(x,t)\,.$$

Thus, (3.4) holds true. Consequently, we see that there exist sequences $\{u^n\}$ and $\{v^n\}$ satisfying (3.3), (3.3)$^+$ and (3.4).

Now, using the order preserving property of $S$ again, we find that $\{u^n\}$ is a nondecreasing sequence. Use (1.3) with $u=\pm u^n$ and $\hat{u}=\hat{f}=\hat{u}_0=0$ to obtain

$$|g_2(x,t,u^n)|_1\leq\|u_0\|_1+|g_1(x,t,u^{n-1})+f|_1\leq\|u_0\|_1+|f|_1+\int_Q g_1(x,t,\tilde{\rho}(h+a))dxdt\,. \tag{3.5}$$

Let $u=\lim_{n\to\infty}u^n$. Since $g_2(x,t,u^n)$ converges monotonously to $g_2(x,t,u)$ for a.e. $(x,t)$ in $Q$, Beppo-Levi's lemma together with (3.5) yields that $g_2(\cdot,\cdot,u^n)$ converges to $g_2(\cdot,\cdot,u)$ in $L^1(Q)$ as $n\to\infty$. On the other hand, since $g_1(\cdot,\cdot,u^n)\leq g_1(\cdot,\cdot,\tilde{\rho}(h+a))\in L^1(Q)$, Lebesgue's convergence theorem yields that $g_1(\cdot,\cdot,u^n)\to g_1(\cdot,\cdot,u)$ and $u^n\to u$ in $L^1(Q)$ as $n\to\infty$. Therefore, passing to the limit in (3.3) yields (3.1). □

## 4. Equations involving measures.

In this section we apply Theorem 3.1 to the problem

$$\begin{aligned}&u\in X\cap L^q(Q)\,,\quad u^+\in L^p(Q)\,,\\&u_t-\Delta u-(u^+)^p+u|u|^{q-1}=\nu\qquad\text{in }\mathscr{D}'(Q)\,.\\&\operatorname{ess\,lim}_{t\to+0}u(\cdot,t)=\mu\qquad\text{in }m_b(\Omega)\,,\end{aligned}\tag{4.1}$$

where $\mu\in m_b(\Omega)$, $\nu\in m_b(Q)$ and $p,q>1$. In the case whepe the term $u|u|^{q-1}$ disappears in (4.1) this problem has been treated by many authors (e.g. [4], [5], [10], [14[, [17], [18]). In the case where the term $(u^+)^p$ disappears in (4.1) it was considered in [6], [7], etc..

To mention the results about (4.1) let $D=\Omega\times(-T,T)$ and recall that $W_q^{-2,-1}(D)$ denotes the dual space of $\mathring{W}_{q'}^{2,1}(D)$, where $q>1$ and $q'=q/(q-1)$.

THEOREM 4.1. *Let $p>q>1$, $\mu\in m_b(\Omega)$ and $\nu\in m_b(Q)$. Suppose that one of the following conditions is satisfied:*

(a) *$p<(N+2)/N$ and*

$$\begin{aligned}&\mu=\mu_1+\mu_2\,,\quad\nu=\nu_1+\nu_2\\&\mu_1\in L^1(\Omega)\,,\quad\nu_1\in L^1(Q)\\&\tilde{\nu}_2+\mu_2\otimes\delta\in W_q^{-2,-1}(D)\end{aligned}\tag{4.2}$$

*where $\tilde{\nu}_2$ is the measure on $D$ such that $\tilde{\nu}_2(E)=\nu_2(E\cap Q)$ for all measurable subset $E$ of $D$,* *and $\delta$ is the Dirac measure at the origin on* $(-T, T)$.

(b) $p>(N+2)/N$, $\mu\in L^r(\Omega)$, $\nu\in L^r(Q)$ *and* $r>p(N-1)/2$.

(c) $p=(N+2)/N$, $\mu\in L^r(\Omega)$, $\nu\in L^r(Q)$ *and* $r>1$.

*Then,* (4.1) *has a local solution on* $[0, T']$ *with some* $T'\in(0, T]$.

THEOREM 4.2. *Let* $q\geq p>1$. *Let* $\mu\in m_b(\Omega)$ *and* $\nu\in m_b(Q)$ *satisfy* (4.2). *Then,* (4.1) *has a global solution on* $[0, T]$.

*Remark.* Condition (4.2) can be characterized by terms of capacities (see Proposition 4.3 below). Thus Theorems 4.1 and 4.2 extend some results of [5], [6], [7], [17], [18], and offer a unified treatment for problems of the type (4.1).

*Proof of Theorem* 4.1. We first assume that (a) holds. Let $V=L(\mu_2, \nu_2)$. (4.2) implies that $V\in L^q(Q)$. Indeed,

$$|V|_q\leq C|\tilde{\nu}_2+\mu_2\otimes\delta|_{-2,-1,q} \tag{4.3}$$

where $|\cdot|_{-2,-1,q}$ denotes the norm of $W_q^{-2,-1}(D)$. We set

$$g_1(x, t, r)=k_1(r+V(x, t)),$$
$$g_2(x, t, r)=k_2(r+V(x, t))-k_2(V(x, t)),$$
$$u_0=\mu_1 \quad\text{and}\quad f=\nu_1-k_2(V(\cdot, \cdot))$$

for a.e. $(x, t)$ in $Q$ and $r$ in $R$; where $k_1(r)=(r^+)^p$ and $k_2(r)=r|r|^{q-1}$. Then, $u$ is a solution of (4.1) if and only if $v=u+V$ is a solution of (3.1) with those $g_1$, $g_2$, $u_0$ and $f$. Therefore we must check Conditions (H1)-(H5) in the previous section. However, (H1) and (H2) are obvious. (H4) and (H5) follow from (4.3). To show (H3) we set

$$\phi=\mu_2^+ \quad\text{and}\quad \psi=\nu_2^+ .$$

Set $w=L(\phi, \psi)$ and $h=L(u_0^++\phi, f^++\psi)$. Noting that $V\leq w\leq h$, we have

$$g_1(x, t, r-w)\leq g_1(x, t, r-V)=r^p \qquad \text{for } r\geq 0,$$

and

$$g_1(x, t, \alpha h)\leq C(h^p+(V^+)^p)\leq Ch^p \qquad \text{for } \alpha>0 .$$

Since $h\in L^p(Q)$ whenever $p<(N+2)/N$ (see Lemma 1.1 (c)), (H3)-(i) and (iii) hold with $\gamma=p$. To see (H3)-(ii) let $\theta\in L^\infty(Q)^+$ and $\theta$ have a compact support in $Q$ and let $\xi(x, t)=\hat{\xi}(x, T-t)$ for $(x, t)\in Q$ where $\hat{\xi}=L(0, \theta)$. If

$$2-(\beta^{-1}-\alpha^{-1})(N+2)\geq 0 \tag{4.4}$$

holds, then by the embedding theorem (cf. [11]) we have

$$|\xi|_\alpha\leq C|\xi|_{2,1,\beta}\leq C|\theta|_\beta . \tag{4.5}$$

Put $\zeta=\theta^{p'}\xi^{1-p'}$. If, furthermore,

$$\beta<p', \qquad \beta(p'-1)/(p'-\beta)\leq\alpha\leq\infty \tag{4.6}$$

holds, then by Hölder's inequality we have

$$|\theta|_\beta \leq \left\{\int_Q \zeta dxdt\right\}^{1/p'} \left\{\int_Q \xi^{\beta(p'-1)/(p'-\beta)} dxdt\right\}^{(p'-\beta)/p'\beta}$$
$$\leq C|\zeta|_1^{1/p'}|\xi|_\alpha^{(p'-1)/p'} \leq C|\zeta|_1^{1/p'}|\theta|_\beta^{(p'-1)/p'}$$

and hence

(4.7) $$|\theta|^\beta \leq C|\zeta|_1 .$$

Take $\alpha=\infty$. Since $p<(N+2)/N$, it is possible to choose such a $\beta$ satisfying (4.4) and (4.6). Hence, (4.5) and (4.7) give

$$\|\xi(\cdot,0)\|_\infty+|\xi|_\infty \leq C|\theta|_{\hat\beta} \leq C\, m(Q)^\tau|\theta|_\beta \leq C\, m(Q)^\tau|\psi|_1 \qquad \text{for } \beta>\hat\beta\geq(N+2)/2 ,$$

where $\tau=(\beta-\hat\beta)/\beta\hat\beta$. Since the conjugate function of $g(r)=r^p$ is $g^*(r)=(p-1)(r/p)^{p'}$, the definition of $N_{g,T}$ yields

(4.8) $$N_{g,T}(u_0^+ +\phi, f^+ +\psi) \leq C\, m(Q)^\tau\left\{\int_\Omega d(u_0^+ +\phi)+\int_Q d(f^+ +\psi)\right\} \leq C\, m(Q)^\tau(\|\mu_1\|_1+1) .$$

Note here that the constants $C$ appearing in the above inequalities do not depend on $\mu_1$. Thus, (H3)-(ii) holds if $T>0$ is sufficiently small. Consequently Theorem 3.1 guarantees that (4.1) has a solution on $[0, T']$ with some $T'\in(0, T]$.

Next, let us consider the case where (b) or (c) holds. In this case we set

$$g_1(x,t,r)=(r^+)^p , \quad g_2(x,t,r)=r|r|^{q-1} ,$$
$$u_0=\mu \quad \text{and} \quad f=\nu .$$

Since (H1), (H2), (H4) and (H5) are clear, we shall show that (H3) holds with $\phi=\psi=0$ and $\gamma=p$. For this end we estimate the function $h=L(u_0^+, f^+)$. We know ([17], [10, Lemma]) that

$$h(t)=e^{-t\Delta}u_0^+ +\int_0^t e^{-(t-s)\Delta}f^+(s)ds ,$$
$$\|e^{-t\Delta}a\|_\alpha \leq Ct^{-N(\beta^{-1}-\alpha^{-1})/2}\|a\|_\beta \qquad \text{for } \alpha\geq\beta\geq1 ,$$
$$\int_0^t \|e^{-s\Delta}a\|_\alpha^\alpha ds \leq C\|a\|_\beta^\alpha \qquad \text{for } \beta=N\alpha/(N+2) .$$

Combining these facts with Young's inequality leads to

(4.9) $$|h|_{p\alpha'} \leq C(\|u_0^+\|_\eta+|f^+|_\eta) \qquad \text{with } \eta=Np\alpha'/(N+2) ,$$

provided

(4.10) $$Np\alpha'-N-2>0 , \qquad \alpha'=\alpha/(\alpha-1)$$

holds. Therefore, if $\alpha$ and $\beta$ satisfy (4.4), (4.6) and (4.10), then we have

$$\int_\Omega \xi(\cdot,0)du_0^+ +\int_Q \xi df^+ =\int_Q h(-\xi_t-\Delta\xi)dxdt=\int_Q h\theta dxdt$$
$$=\int_Q h\zeta^{1/p'}\xi^{1/p}dxdt \leq |h|_{p\alpha'}|\zeta|_1^{1/p'}|\xi|_\alpha^{1/p} \leq C(\|u_0^+\|_\eta+|f^+|_\eta)|\zeta|_1$$

which implies that for $r>\eta$

$$N_{g,T}(u_0^+, f^+)\leq C\, m(Q)^{(r-\eta)/r\eta}(\|u_0^+\|_r+|f^+|_r)\ . \tag{4.11}$$

Now we set $\alpha=(N+2)(p-1)/(Np-N-2)$ and $\beta=(N+2)(p-1)/((N+2)p-N-4)$ if $p>(N+2)/N$; and an arbitrary $\alpha>1$ and $\beta=(N+2)\alpha/(2\alpha+N+2)$ if $p=(N+2)/N$. Then (4.4), (4.6) and (4.10) hold for a certainty. Therefore it follows from (4.8) and (4.11) that (H3) holds with $\phi=\psi=0$ and $\gamma=p$ if $T>0$ is sufficiently small.

*Proof of Theorem* 4.2. In this case we set

$$g_1(x,t,r)=\tilde{k}_1(r+V(x,t))\ ,$$
$$g_2(x,t,r)=\tilde{k}_2(r+V(x,t))-\tilde{k}_2(V(x,t))\ ,$$
$$u_0=\mu_1 \quad\text{and}\quad f=\nu_1-\tilde{k}_2(V(\cdot,\cdot))$$

for a.e. $(x,t)\in Q$ and $r\in R$; where $V=L(\mu_2,\nu_2)$ and

$$\tilde{k}_1(r)=\begin{cases}(r^+)^p & \text{if } r\leq 1\ ,\\ \ 1 & \text{if } r>1\ ,\end{cases}$$
$$\tilde{k}_2(r)=\begin{cases}r|r|^{q-1} & \text{if } r\leq 1\ ,\\ r^q-r^p+1 & \text{if } r>1\ .\end{cases}$$

We also see that $u$ is a solution of (4.1) if and only if $v=u+V$ is a solution of (3.1) with those $g_1, g_2, u_0$ and $f$. Now, (H1), (H2), (H4) and (H5) are obvious. To see (H3) set

$$\phi=\mu_2^+\ ,\quad \psi=\nu_2^+\quad\text{and}\quad \gamma=\min\{p,(N+1)/N\}\ .$$

Then we have that $g_1(x,t,r-w)\leq r^\gamma$ for $r\geq 0$, which implies (H3)-(i). (H3)-(iii) is a direct consequence of the fact that $g_1(x,t,r)\leq 1$ on $Q\times R$. Since $\gamma<(N+2)/N$, the same manner as in the proof of Theorem 4.1 also yields (4.8) with $g(r)=r^\gamma$. Therefore there exists a solution $v_1$ of (3.1) and hence a solution $u_1$ of (4.1) on $[0,T']$ for some $T'\in(0,T]$.

Next, consider the problem (4.1) where $\mu$ and $\nu$ are replaced by $\hat{\mu}=u_1(\cdot,T')$ and $\hat{\nu}=\nu(\cdot,\cdot,+T')$, respectively. Condition (4.2) is clearly satisfied by putting $\hat{\mu}_1=u_1(\cdot,T')\in L^1(\Omega)$ and $\hat{\mu}_2=0$. Therefore there exists a solution $u_2$ of (4.1) with $u_2(\cdot,0)=u_1(\cdot,T')$ on $[0,T'']$ for some $T''\in(0,T-T']$. Define $u$: $\Omega\times(0,T'+T'']\to R$ by $u(\cdot,t)=u_1(\cdot,t)$ for $t\in(0,T']$ and $u(\cdot,t)=u_2(\cdot,t-T')$ for $t\in[T',T'+T'']$. It is not hard to see that $u$ is a solution of (4.1) on $[0,T'+T'']$. We here remark that $T''$ is determined by (H3)-(ii) only, that is, by the condition that $N_{g,T''}(\hat{\mu}^+, f^++\hat{\nu}_2^+)<1$ (Note that $u_0=\hat{\mu}, f=\hat{\nu}_1-\tilde{k}_2(V), \phi=\hat{\mu}_2^+=0, \psi=\hat{\nu}_2^+$ and $g(r)=r^\gamma$). But, (4.8) gives

$$N_{g,T''}(\hat{\mu}^+, f^++\hat{\nu}_2^+)\leq C\,\mathrm{m}(\Omega\times(0,T''))^\tau(\|\hat{\mu}_1\|_1+1)$$

with some constant $C$ which depends on only $\Omega$, $T$, $p$, $q$, $N$, and $\nu$. Moreover, (1.3) gives

$$\|v_1(\cdot,T')\|_1\leq\|u_0\|_1+|f+g_1(\cdot,\cdot,v_1)|_1\leq\|\hat{\mu}_1\|_1+|\nu_1|_1+|V|_q^q+2\,m(Q)$$

where $v_1=u_1+V$. Thus we obtain that $N_{g,T''}(\hat{\mu}^+, f^++\hat{\nu}_2^+)\leq C\, m(\Omega\times(0,T''))^\tau$ for some constant $C$ which depends on only $\Omega$, $T$, $p$, $q$, $\mu$ and $\nu$. This implies that $T''$ is determined by given data only. Therefore we can extend $u$ to $[0,T]$. $\square$

Finally, recall the definitions of capacities with respect to the spaces $W_p^{2,1}(R^{N+1})$ and

$W^{\alpha,p}(R^N)$. Let $E$ be a subset of $R^{N+1}$. If $E$ is compact, we set

$$c_{2,1,p}(E)=\inf\{|v|_{2,1,p}^p;\ v\in C_0^\infty(R^{N+1}), v\geq 1 \text{ on } E\}.$$

If $E$ is open, we set

$$c_{2,1,p}(E)=\sup\{c_{2,1,p}(K);\ K\subset E, K \text{ is compact}\}.$$

If $E$ is an arbitrary subset, we set

$$c_{2,1,p}(E)=\inf\{c_{2,1,p}(G);\ E\subset G, G \text{ is open}\}.$$

$c_{2,1,p}$ is called a $W_p^{2,1}$ capacity on subsets of $R^{N+1}$. Similarly, we can define a $W^{\alpha,p}$-capacity on subsets of $R^N$ by using the norm $\|\cdot\|_{\alpha,p}$ in $W^{\alpha,p}(R^N)$. We refer to [5], [13] and [2] for the properties of the capacities and the relation between Hausdoff measure and capacity.

Using these concepts we can characterize (4.2) as follows:

PROPOSITION 4.3. *Let* $q>1, q'=q/(q-1), \mu\in m_b(\Omega)$ *and* $\nu\in m_b(Q)$. *Then,* (4.2) *holds if and only if the following condition holds*:

$$\begin{aligned} &E\subset R^{N+1} \text{ and } c_{2,1,q'}(E)=0 \text{ implies } |\nu|(E)=0,\\ &F\subset R^N \text{ and } c_{2/q,q'}(E)=0 \text{ implies } |\mu|(F)=0. \end{aligned} \tag{4.12}$$

*Proof*. This is essentially proved in [6]. For simplicity set $\kappa=\tilde{\nu}+\mu\otimes\delta$ where $\tilde{\nu}$ is the extension of $\nu$ to $D$ by 0. We know ([6, Proposition 2.3]) that (4.12) is equivalent to

$$E\subset D \text{ and } c_{2,1,q'}(E)=0 \text{ implies } |\kappa|(E)=0. \tag{4.13}$$

Therefore, it sufficies to show that (4.13) is equivalent to

$$\kappa=\kappa_1+\kappa_2,\quad \kappa_1\in L^1(D),\quad \kappa_2\in W_q^{-2,-1}(D). \tag{4.14}$$

It is a direct consequence of [6, Proposition 3.1] that (4.14) implies (4.13). Conversely, we show that (4.13) implies (4.14). We may assume that $\kappa\geq 0$. Otherwise consider the Jordan decomposition $\kappa=\kappa^+-\kappa^-$. Assume that (4.13) holds. By [6, Proposition 3.2] there exists a sequence $\{\sigma_n\}$ in $m_b(D)^+$ such that $\sigma_n\in W_q^{-2,-1}(D)$, supp $\sigma_n$ is compact and $\sum_{n=1}^\infty\sigma_n=\kappa$ in $m_b$(D). Let $\rho_m$ be a mollifier on $R^{N+1}$. Observe that

$$|\rho_m*\sigma_n-\sigma_n|_{-2,-1,q}\leq|\rho_m*v_n-v_n|_q\to 0 \quad \text{as} \quad m\to\infty.$$

Here, $(v_n)_t-\Delta v_n=\sigma_n$ in $D$, $v_n(\cdot,-T)=0$ in $\Omega$, $v_n=0$ on $\partial\Omega\times(-T,T)$. Hence there exist $\kappa_2\in W_q^{-2,-1}(D)$ and a subsequence $\{m_n\}$ satisfying $\sum_{n=1}^\infty(\sigma_n-\rho_{m_n}*\sigma_n)$ is absolutely convergent to $\kappa_2$ in $W_q^{-2,-1}(D)$. Also since

$$\sum_{n=1}^\infty|\rho_{m_n}*\sigma_n|_1\leq\sum_{n=1}^\infty\sigma_n(D)=\kappa(D)<+\infty,$$

there exists $\kappa_1\in L^1(D)$ satisfying $\sum_{n=1}^\infty\rho_{m_n}*\sigma_n$ is absolutely convergent to $\kappa_1$ in $L^1(D)$. Therefore we have

$$\kappa_2=\sum_{n=1}^\infty(\sigma_n-\rho_{m_n}\sigma_n)=\kappa-\kappa_1.$$

□

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