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On Some Special Finsler Spaces

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Finsler spaces have been studied by many mathematicians and phisicists, especially in Japan. In the present paper, we shall consider some special Finsler spaces such as $R3$-like ones, of scalar curvature, of perpendicular scalar curvature, of Rp-scalar curvature, of Hp-scalar curvature, with $F_h{}^i{}_{jk}=0$, $*P$-Finsler spaces, Landsberg spaces etc. and investigate relationships among them.

§1. Preliminaries

Let F_n be an n-dimensional Finsler space with a fundamental function $F(x^i, y^i)$[1]. Here, we assume that $F(x,y)$ satisfies the following conditions: 1) $F(x,y)$ is a positively homogeneous function with respect to y^i, that is, $F(x,\lambda y)=\lambda F(x,y)$ for $\lambda>0$; 2) $F(x,y)$ is positive for $y^i\neq 0$; 3) the *fundamental tensor* $g_{ij}:=(1/2)\partial^2F^2/\partial y^i\partial y^j$[2] is positive definite, that is, $g_{ij}X^iX^j>0$ for any variables $X^i\neq 0$ (in detail, see the paper [7][3] appeared in this journal, or [9], [14] etc.).

A hypersurface of F_n defined by the equation

$$F(x,y)=1\ ,$$

where the point $x=(x^i)$ is fixed and y^i are variables, is called the *indicatrix*. We denote by p· the projection on the indicatrix, for example, for a tensor T^i_{jk} of type (1,2), we can see

$$\begin{aligned}\mathrm{p}\cdot T^i_{jk}=h^i_aT^a_{bc}h^b_jh^c_k=T^i_{jk}-F^{-1}(l^iT^0_{jk}+l_jT^i_{0k}+l_kT^i_{j0})+F^{-2}(l^il_jT^0_{0k}+l^il_kT^0_{j0}+l_jl_kT^i_{00})\\ -F^{-3}(l^il_jl_kT^0_{00})\ ,\end{aligned}$$

where $h^i_a:=\delta^i_a-l^il_a$, $l_j:=\partial F/\partial y^j$, $l^i:=g^{ij}l_j=F^{-1}y^i$, δ^i_j is the Kronecker delta, g^{ij} are the reciprocal components of g_{ij} in the matrix (g_{ij}) and the index 0 means the contraction by y, e.g., $T^i_{j0}=T^i_{jk}y^k$, $T^0_{jk}=T^i_{jk}y_i$, $y_i:=y^jg_{ij}$. The tensor $h_{ij}:=g_{im}h^m_j$ is called the *angular metric tensor*. A tensor T satisfying $\mathrm{p}\cdot T=T$ is called an *indicatric* tensor. h^i_j or h_{ij} is indicatric.

We use two kinds of covariant derivatives due to Cartan, that is, for a tensor T^i_j of type (1,1)

$$\text{(1.1)}\qquad \begin{aligned}&\text{a)}\quad T^i_{j/k}:=d_kT^i_j+{}^*\Gamma^i_{hk}T^h_j-{}^*\Gamma^h_{jk}T^i_h\ ,\\ &\text{b)}\quad T^i_{j/(k)}:=T^i_{j(k)}+C^i_{hk}T^h_j-C^h_{jk}T^i_h\ ,\end{aligned}$$

where

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1) Latin indices run over $1,2,\dots,n$. We may use $F(x, y)$ or merely F instead of $F(x^i, y^i)$.
2) A:=B or B=:A means that A is defined by B. Also, we apply the Einstein's summation convention.
3) Numbers in square brackets refer to the references at the end of this paper.

$$d_k:=\partial/\partial x^k-G^i_k\partial/\partial y^i\ ,\quad G^i_k:=\partial G^i/\partial y^k\ ,\quad G^i:=\frac{1}{2}\gamma^i_{jk}y^jy^k\ ,\quad {}_{(k)}:=\partial/\partial y^k\ ,$$

$$\gamma^i_{jk}:=\frac{1}{2}g^{ih}(\partial g_{hj}/\partial x^k+g_{hk}/\partial x^j-\partial g_{jk}/\partial x^h)\ ,\quad {}^*\Gamma^i_{jk}:=\frac{1}{2}g^{ih}(d_kg_{hj}+d_jg_{hk}-d_hg_{jk})\ ,$$

$$C^i_{hk}:=g^{ij}C_{hjk}\ ,\quad C_{hjk}:=\frac{1}{2}\partial g_{hj}/\partial y^k\ .$$

Then, the curvature and torsion tensors are defined as follows:

$$
(1.2)\quad
\begin{aligned}
&\text{a)}\ R_h{}^i{}_{jk}:=(d_k{}^*\Gamma^i_{hj}+{}^*\Gamma^m_{ij}{}^*\Gamma^i_{mk}-j|k)+C^i_{hm}H^m_{jk}\ ,\quad R_h{}^m{}_{jk}g_{mi}=:R_{hijk}\ ,\\
&\text{b)}\ P_{hijk}:=C_{ijk/h}+C^m_{hj}P_{mik}-h|i\ ,\\
&\text{c)}\ S_{hijk}:=-C^m_{hj}C_{mik}-j|k\ ,\quad S_{hmjk}g^{mi}=:S_h{}^i{}_{jk}\ ,\\
&\text{d)}\ H_h{}^i{}_{jk}:=d_kG^i_{hj}+G^m_{hj}G^i_{mk}-j|k=H^i_{jk(h)}\ ,\quad H_h{}^m{}_{jk}g_{mi}=:H_{hijk}\ ,\\
&\text{e)}\ H^i_{jk}:=d_kG^i_j-j|k=R_0{}^i{}_{jk}=H_0{}^i{}_{jk}\ ,\quad H^i_{0k}=:H^i_k\ ,\\
&\text{f)}\ P^i_{jk}:=C^i_{jk/0}\ ,\quad P^m_{jk}g_{mi}=:P_{jik}\ ,
\end{aligned}
$$

where $G^i_{jk}:=\partial G^i_j/\partial y^k$ and $-j|k$ means the interchange of indices j, k in the foregoing terms and subtraction. S_{hijk}, P_{hijk} and R_{hijk} are called the *first, second* and *third curvature tensors of Cartan*, respectively. On the other hand, H_{hijk} is called the *Berwald curvature tensor*.

It is known (e.g., [17], (1.5)) that the third curvature tensor of Cartan and the Berwald curvature tensor are related by the following relation:

$$(1.3)\qquad R_{hijk}=\frac{1}{2}(H_{hijk}-h|i)-Q_{hijk}\ ,$$

where $Q_{hijk}:=P^m_{hj}P_{mik}-j|k$.

Also, we know that the Berwald curvature tensor satisfies the following identities:

$$
(1.4)\quad
\begin{aligned}
&\text{a)}\ H_{hijk}-H_{jkhi}=(H^m_{hj}C_{mik}+H^m_{ik}C_{mhj}+P_{hij/k}-j|k)-H^m_{jk}C_{mhi}+H^m_{hi}C_{mjk}\\
&\qquad\qquad -P_{jkh/i}+P_{jki/h},\\
&\text{b)}\ H_{hiok}=H_{khi}-H^m_hC_{mki}+H^m_iC_{mkh}-H^m_kC_{mhi}-P_{hik/o},\\
&\text{c)}\ H_{oijk}=H_{ijk}=-H_{iojk},\\
&\text{d)}\ H_{hook}=-H_{hok}\ .
\end{aligned}
$$

§ 2. An *R*3-like Finsler space

M. Matsumoto [8] showed that in a three-dimensional Finsler space the third curvature tensor of Cartan is always expressed by

$$(2.1)\qquad R_{hijk}=g_{hj}L_{ik}+g_{ik}L_{hj}-j|k\ ,$$

where $L_{ik}=(R_{ik}-(1/2)rg_{ik})/(n-2)$, $R_{ik}:=R_i{}^m{}_{km}$, $r:=g^{rs}R_{rs}/(n-1)$. So, we shall give the following

Definition 2.1. If the curvature tensor R_{hijk} in a Finsler space $F_n(n>3)$ has the form (2.1), then the space is called an *R*3-*like* Finsler space.

Let us construct a tensor C_{hijk} formally from the curvature tensor R_{hijk} by the same expression as that of the conformal curvature tensor in a Riemannian space, that is,

$$C_{hijk} := R_{hijk} - (g_{hj}R_{ik} + g_{ik}R_{hj} - rg_{hj}g_{ik} - j|k)/(n-2) .$$

In this case, H. Izumi and T. N. Srivastava ([3], Theorem 3.3) showed

Theorem 2.1. *An R3-like Finsler space is characterized by* $C_{hijk}=0$.

Now, we shall decompose the tensor L_{ik} in an $R3$-like Finsler space by the idea of indicatrization (for this idea, see [6], [3]) as follows:

$$L_{ik} = m_{ik} + a_i l_k + l_i b_k + c l_i l_k , \tag{2.2}$$

where $m_{ik} := \mathrm{p}\cdot L_{ik} = m_{ki}$ (cf. [3], (3.9)b)), $a_i := F^{-1}\mathrm{p}\cdot L_{io}$, $b_k := F^{-1}\mathrm{p}\cdot L_{ok}$, $c := F^{-2}L_{oo}$. Accordingly, taking account of (1.2)e), we get

$$\text{(2.3)} \quad \begin{aligned} &\text{a)} \quad H^i_{jk} = F[l_j(m^i_k + ch^i_k) + b_j h^i_k] - j|k , \\ &\text{b)} \quad H^i_k = F^2(m^i_k + ch^i_k) , \end{aligned}$$

where $m^i_k := g^{ij}m_{jk}$. Then, the following identities are known [3]:

$$\text{(2.4)} \quad \begin{aligned} &\text{a)} \quad \mathrm{p}\cdot R_{hij/k} + P^m_{hj}P_{mik} + Fb_k C_{hij} + h_{ij}(-m_{hk} + b_{hk} + ch_{hk}) - j|k = 0 , \\ &\text{b)} \quad 2P^m_{hj}P_{mik} - 2ch_{hj}h_{ik} + h_{hj}(m_{ik} - b_{ik}) + h_{ik}(m_{hj} - b_{hj}) - j|k = 0 , \end{aligned}$$

where $b_{ik} := F\mathrm{p}\cdot b_{k(i)} = b_{ki}$ (cf. [3], Lemma 5.4). These identities will be used later.

§ 3. A Finsler space of scalar curvature

Definition 3.1. Let $X=(X^i)$ be a vector of a Finsler space $F_n (n>2)$ at a point $x=(x^i)$. The quantity $K(x, y, X)$ at (x, y) given by

$$K(x, y, X) = \frac{R_{hijk}y^h X^i y^j X^k}{(g_{hj}g_{ik} - g_{hk}g_{ij})y^h X^i y^j X^k}$$

is called the (*sectional*) *curvature* at (x, y) with respect to X. Then, if $K(x, y, X)$ is independent of X at any (x, y), then the space is said to be *of scalar curvature K*. Especially, if K is constant, then the space is said to be *of constant curvature.*

In the above R_{hijk} can be replaced by H_{hijk}, because $R_{oiok} = H_{oiok}$ holds good.

The following important facts are known:

Theorem 3.1 ([14], [11], [16]). *A Finsler space of scalar curvature K is characterized by any one of the following equations*:

$$\text{(3.1)} \quad \begin{aligned} &\text{a)} \quad H^i_k = F^2 K h^i_k , \\ &\text{b)} \quad H^i_{jk} = F\left(Kl_j + \frac{1}{3}K_j \right) h^i_k - j|k , \\ &\text{c)} \quad H_h{}^i{}_{jk} = \left[l_h \left(Kl_j + \frac{1}{3}K_j \right) + \left(Kh_{hj} + \frac{2}{3}K_h l_j \right) + \frac{1}{3}K_{hj} \right] h^i_k + l^i \left(Kl_k + \frac{1}{3}K_k \right) h_{hj} \\ &\qquad + \frac{1}{3}h^i_h l_j K_k - j|k , \end{aligned}$$

where $K_j := FK_{(j)}, K_{hj} := F\text{p}\cdot K_{j(h)} = K_{jh}$.

Theorem 3.2 (e.g., [14], p. 123). *If the curvature K in a Finsler space of scalar curvature is independent of y, then K is constant.*

§ 4. A Finsler space of perpendicular scalar curvature

Analogously to a Finsler space of scalar curvature, we shall give the following

Definition 4.1 ([4], [5]). Let $X=(X^i)$ and $Y=(Y^i)$ be two independent vectors of a Finsler space $F_n(n>3)$ at a point $x=(x^i)$. The quantity $R(x, y, \text{p}\cdot X, \text{p}\cdot Y)$ at (x, y) given by

$$R(x,y,\text{p}\cdot X,\text{p}\cdot Y)=\frac{R_{hijk}(\text{p}\cdot X^h)(\text{p}\cdot Y^i)(\text{p}\cdot X^j)(\text{p}\cdot Y^k)}{(g_{hj}g_{ik}-g_{hk}g_{ij})(\text{p}\cdot X^h)(\text{p}\cdot Y^i)(\text{p}\cdot X^j)(\text{p}\cdot Y^k)}, \tag{4.1}$$

is called a *perpendicular sectional curvature* at (x, y) with respect to X and Y. In addition, if $R(x, y, \text{p}\cdot X, \text{p}\cdot Y)$ is independent of X and Y at any (x, y), then the space is said to be *of perpendicular scalar curvature* (abbreviated *of p-scalar curvature*).

A characterization of a Finsler space of p-scalar curvature and the curvature tensor R_{hijk} of this space are given respectively by the following theorems [5]:

Theorem 4.1. *A Finsler space of p-scalar curvature is characterized by*

$$\text{p}\cdot R_{hijk}=Rh_{hj}h_{ik}+\frac{1}{2}(Z^m_{hj}C_{mik}+Z^m_{ik}C_{mhj})-j|k\ , \tag{4.2}$$

where $Z^m_{hj} := \text{p}\cdot H^m_{hj}$.

Theorem 4.2. *The curvature tensor R_{hijk} of a Finsler space of p-scalar curvature has the form*

$$\begin{aligned}R_{hijk}=&F^{-1}(l_h g_{im}H^m_{jk}-l_i g_{hm}H^m_{jk}+l_j g_{km}H^m_{hi}-l_k g_{jm}H^m_{hi}\\&-F^{-2}(l_h l_j g_{im}H^m_k+l_i l_k g_{hm}H^m_j-j|k)-F^{-1}(C_{mhj}H^m_i l_k+C_{mik}H^m_h l_j-j|k)\\&+\left[Rh_{hj}h_{ik}+\frac{1}{2}(Z^m_{hj}C_{mik}+Z^m_{ik}C_{mhj})-j|k\right].\end{aligned} \tag{4.3}$$

A Finsler space of p-scalar curvature and a Finsler space of scalar curvature are independent of each other. So, we shall give the following

Definition 4.2. If a Finsler space of scalar curvature is at the same time of p-scalar curvature, then the space is called a Finsler space *of s-ps curvature.*

It is proved that the curvature tensor R_{hijk} of a Finsler space of s-ps curvature has the form similar to (2.1) of that in an R3-like Finsler space, namely we have

Theorem 4.3. *The curvature tensor R_{hijk} of a Finsler space of s-ps curvature has the form*

$$R_{hijk}=h_{hj}M_{ik}+h_{ik}M_{hj}-j|k\ , \tag{4.4}$$

where $M_{ik}:=\frac{1}{2}Rh_{ik}+\frac{1}{3}(K_il_k+l_iK_k)+Kl_il_k$.

Proof. Substituting (3.1)a), b) into (4.3), we can calculate as follows:

$$\begin{aligned}R_{hijk}=&\left[\left(Kl_hl_j+\frac{1}{3}l_hK_j\right)h_{ik}-\left(Kl_il_j+\frac{1}{3}l_iK_j\right)h_{hk}-j|k\right]\\&+\left[\left(Kl_jl_h+\frac{1}{3}l_jK_h\right)h_{ki}-\left(Kl_kl_h+\frac{1}{3}l_kK_h\right)h_{ji}-h|i\right]\\&-K(l_hl_jh_{ik}+l_il_kh_{hj}-j|k)-FK(C^m_{hj}h_{mi}l_k+C^m_{ik}h_{mh}l_j-j|k)\\&+\left[Rh_{hj}h_{ik}+\frac{1}{6}F\{(K_hh^m_j-h|j)C_{mik}+(K_ih^m_k-i|k)C_{mhj}\}-j|k\right]\\=&\frac{1}{3}(l_hK_jh_{ik}+l_iK_kh_{hj})+\left(Kl_jl_h+\frac{1}{3}l_jK_h\right)h_{ki}+\left(Kl_kl_i+\frac{1}{3}l_kK_i\right)h_{hj}+Rh_{hj}h_{ik}\\&-\frac{1}{6}(K_hC_{jik}-K_jC_{hik}+K_iC_{khj}-K_kC_{ihj})-j|k\\=&h_{ik}\left[\frac{1}{3}(l_hK_j+l_jK_h)+Kl_hl_j+\frac{1}{2}Rh_{hj}\right]+h_{hj}\left[\frac{1}{3}(l_iK_k+l_kK_i)+Kl_il_k+\frac{1}{2}Rh_{ik}\right]-j|k\\=&h_{hj}M_{ik}+h_{ik}M_{hj}-j|k\,.\end{aligned}$$

Q.E.D.

Theorem 4.4 (cf. [3]). *A Finsler space of s-ps curvature is an R3-like Finsler space.*

Proof. Making use of $h_{hj}=g_{hj}-l_hl_j$, we shall rewrite (4.4). Then, we have

$$\begin{aligned}R_{hijk}&=(g_{hj}-l_hl_j)M_{ik}+(g_{ik}-l_il_k)M_{hj}-j|k\\&=g_{hj}M_{ik}+g_{ik}M_{hj}-l_hl_jM_{ik}-l_il_kM_{hj}-j|k\\&=g_{hj}M_{ik}+g_{ik}M_{hj}-\frac{1}{2}R(l_hl_jh_{ik}+l_il_kh_{hj})-\frac{1}{3}(l_hl_jl_iK_k+l_il_kl_hK_j)-j|k\\&=g_{hj}M_{ik}+g_{ik}M_{hj}-\frac{1}{2}R(l_hl_jg_{ik}+l_il_kg_{hj})-j|k\\&=g_{hj}L_{ik}+g_{ik}L_{hj}-j|k\,,\end{aligned}$$

where

$$L_{ik}=M_{ik}-\frac{1}{2}Rl_il_k=\frac{1}{3}(K_il_k+l_iK_k)+\left(K-\frac{1}{2}R\right)l_il_k\,.$$

Q.E.D.

Theorem 4.5 (cf. [3], Theorem 3.6). *An R3-like Finsler space of scalar curvature is a Finsler space of p-scalar curvature, and consequently of s-ps curvature.*

Proof. Since the space is an $R3$-like Finsler space of scalar curvature, comparing (3.1)a) with (2.3)b), we have

$$m_{ik}=mh_{ik}\,,$$

where $m:=m^i_i/(n-1)=K-c$. Thus, from (2.1) we obtain

(4.5) $$\mathrm{p}\cdot R_{hijk}=2mh_{hj}h_{ik}-j|k\ .$$

This means that the space is a Finsler space of p-scalar curvature with $R=2m$ and satisfying $Z^m_{hj}C_{mik}+Z^m_{ik}C_{mhj}-j|k=0$. **Q.E.D.**

§5. A Finsler space of *Rp*-scalar curvature

It may be significant to consider a Finsler space satisfying the form (4.5).

Definition 5.1 [5]. A Finsler space $F_n(n>2)$ satisfying the condition

(5.1) $$\mathrm{p}\cdot R_{hijk}=q(h_{hj}h_{ik}-h_{hk}h_{ij})$$

is called a Finsler space *of Rp-scalar curvature* and q is called the *Rp-scalar curvature.*
Evidently, we have

Theorem 5.1. *A Finsler space $F_n(n>3)$ of Rp-scalar curvature is of p-scalar curvature.*

The following theorem is very essential and important:

Theorem 5.2. *A Finsler space of s-ps curvature is of Rp-scalar curvature.*

Proof. Since the space in consideration is a Finsler space of scalar curvature, from (3.1)b), we have

$$Z^i_{jk}=\frac{1}{3}FK_jh^i_k-j|k\ ,$$

which leads us to $Z^m_{hj}C_{mik}+Z^m_{ik}C_{mhj}-j|k=0$. Hence, in virtue of Theorem 4.1, we get the theorem. **Q.E.D.**
Moreover, we know the following two theorems.

Theorem 5.3 ([3], Proposition 3.1). *An R3-like Finsler space is of Rp-scalar curvature, if m_{ik} is proportional to h_{ik}.*

Theorem 5.4 ([3], Theorem 3.2). *An R3-like Finsler space of Rp-scalar curvature is of scalar curvature, and consequently of s-ps curvature.*

Combining the above two theorems, we can state

Theorem 5.5. *An R3-like Finsler space is s-ps curvature, if m_{ik} is proportional to h_{ik}.*

§6. A Finsler space of *Hp*-scalar curvature

In the previous section we considered a Finsler space with the third curvature tensor of Cartan of a special form. In this section we consider a Finsler space with the Berwald curvature tensor of a special form.

Definition 6.1. A Finsler space $F_n(n>2)$ satisfying the condition

$$\mathrm{p}\cdot H_{hijk}=k(h_{hj}h_{ik}-h_{hk}h_{ij}) \tag{6.1}$$

is called a Finsler space *of Hp-scalar curvature* and k is called the *Hp-scalar curvature.*

Making use of (1.4), we can obtain the Berwald curvature tensor of a Finsler space of Hp-scalar curvature as follows:

$$\begin{aligned} H_{hijk}=&F^{-1}(l_hH_{ijk}-h|i)-F^{-2}(l_hl_jH_{iok}+l_il_kH_{hoj}-j|k)\\ &+F^{-1}[l_j(H_{khi}-H_h^mC_{mki}+H_i^mC_{mkh}-H_k^mC_{mhi}-P_{hik/o})-j|k]+k(h_{hj}h_{ik}-j|k)\,. \end{aligned} \tag{6.2}$$

Now, we assume that a Finsler space of Hp-scalar curvature is at the same time of scalar curvature. Then, from (3.1)c), we get

$$\mathrm{p}\cdot H_{hijk}=\left(Kh_{hj}+\frac{1}{3}K_{hj}\right)h_{ik}-j|k\,. \tag{6.3}$$

In addition, it is known ([17], (3.6)) that in a Finsler space of scalar curvature the following identity holds good:

$$FKC_{hij}+F^{-1}P_{hij/o}+\frac{1}{3}(K_hh_{ij}+h|i|j)=0\,,$$

where $+h|i|j$ means the cyclic permutations of indices h,i,j in the foregoing term and summation. Thus, using the above identity, (3.1)a), b) and (6.2), we have

Theorem 6.1. *The Berwald curvature tensor of a Finsler space of Hp-scalar curvature and at the same time of scalar curvature has the form*

$$H_{hijk}=h_{hj}N_{ik}+h_{ik}N_{hj}-\frac{1}{3}(K_hh_{ij}+h|i|j)l_k-j|k\,, \tag{6.4}$$

where $N_{ik}=(1/2)kh_{ik}+(1/3)(l_iK_k+l_kK_i)+Kl_il_k$.

Taking account of (6.4) and (1.3), we have

Corollary. *The third curvature tensor of Cartan of a Finsler space of Hp-scalar curvature and at the same time of scalar curvature has the form*

$$R_{hijk}=(h_{hj}N_{ik}+h_{ik}N_{hj}-j|k)-Q_{hijk}\,. \tag{6.5}$$

Next, we consider an $R3$-like Finsler space of Hp-scalar curvature. Operating the projection p$\cdot$ to (2.1), we get, with (2.2) in mind,

$$\mathrm{p}\cdot R_{hijk}=h_{hj}m_{ik}+h_{ik}m_{hj}-j|k\,. \tag{6.6}$$

On the other hand, from (6.1) and (1.3), we have

$$\mathrm{p}\cdot R_{hijk}=k(h_{hj}h_{ik}-j|k)-Q_{hijk}\,. \tag{6.7}$$

Therefore, from the above two equations, we get

$$h_{hj}m_{ik}+h_{ik}m_{hj}-j|k=k(h_{hj}h_{ik}-j|k)-Q_{hijk}\,.$$

Transvecting this equation with h^{hj}, we can see

$$m_{ik}=[(n-2)k-(n-1)m]h_{ik}/(n-3)-Q_{ik}/(n-3)\ ,$$

where $Q_{ik}:=Q_i{}^m{}_{km}$. Hence, by means of Theorem 5.3, we can state

Theorem 6.2. *An R3-like Finsler space of Hp-scalar curvature is of Rp-scalar curvature, if Q_{ik} is proportional to h_{ik}.*

§ 7. A Finsler space with $F_h{}^i{}_{jk}=0$

H. Izumi [2] introduced an interesting tensor $F_h{}^i{}_{jk}$ and T. Sakaguchi [15] investigated a Finsler space $F_n(n>2)$ with $F_h{}^i{}_{jk}=0$. This space is characterized by

$$H_h{}^i{}_{jk}=L_{hj}\delta^i_k+g_{hj}L^i{}_k-h^i_hL_{jk}-j|k\ , \tag{7.1}$$

where

$$L_{hj}=\left[(n-1)H_{hj}-\frac{1}{2}g^{rs}H_{rs}g_{hj}+(H_{mh}-H_{hm})l^ml_j\right]\Big/(n-1)(n-2)\ ,$$
$$L^i{}_k:=g^{im}L_{mk}\ ,\quad H_{hj}:=H_h{}^m{}_{jm}\ .$$

In this space, T. Sakaguchi [15] proved the following

Theorem 7.1. *If a Finsler space with $F_h{}^i{}_{jk}=0$ is at the same time of scalar curvature, then the space is a Finsler space of constant curvature.*

When a Finsler space of scalar curvature is replaced by a Finsler space of p-scalar curvature in the above theorem, we have

Theorem 7.2. *If a Finsler space with $F_h{}^i{}_{jk}=0$ is at the same time of p-scalar curvature, then the space is of Rp-scalar curvature.*

Proof. From (7.1), it is easy to see that

$$H^i_{jk}=H_o{}^i{}_{jk}=L_{oj}\delta^i_k+y_jL^i{}_k-j|k\ ,$$

which implies $Z^i_{jk}=(\mathrm{p}\cdot L_{0j})h^i_k-j|k$, hence $Z^m_{hj}C_{mik}+Z^m_{ik}C_{mhj}-j|k=0$. Consequently, using Theorem 4.1, we have the theorem. **Q.E.D.**

Now, let us consider the decomposition of the tensor L_{ik} in (7.1), that is,

$$L_{ik}=m_{ik}+a_il_k+b_kl_i+cl_il_k\ .$$

Substituting this decomposition into (7.1), we obtain

$$H_{hijk}=[l_h\{l_j(m_{ik}+ch_{ik})+b_jh_{ik}\}+h_{hj}m_{ik}+h_{ik}a_hl_j-h|i]+h_{hi}l_j(a_k-b_k)-j|k\ , \tag{7.2}$$

from which, we can see

$$\mathrm{p}\cdot H_{hijk}=(h_{hj}m_{ik}-h|i)-j|k\ . \tag{7.3}$$

By the way, we know

Theorem 7.3 ([15], Theorem 4.4). *A Finsler space with* $F_h{}^i{}_{jk}=0$ *is a Finsler space of constant curvature, if* m_{ik} *is proportional to* h_{ik}.

Here, suppose that a Finsler space with $F_h{}^i{}_{jk}=0$ is of *Hp*-scalar curvature. From (6.1) and (7.3), we have

$$m_{ik}=[(n-2)k-(n-1)m]h_{ik}/(n-3)\ . \tag{7.4}$$

Therefore, by means of Theorem 7.3, we have

Theorem 7.4. *If a Finsler space with* $F_h{}^i{}_{jk}=0$ *is of Hp-scalar curvature, then the space is a Finsler space of constant curvature.*

§ 8. Other special Finsler spaces

Definition 8.1 [1]. A Finsler space satisfying the condition

$$*P^i_{jk}:=P^i_{jk}-\lambda C^i_{jk}=0 \tag{8.1}$$

is called a **P-Finsler space*.
For this space, H. Izumi [1] studied in detail.

Definition 8.2. A Finsler space satisfying the condition

$$P^i_{jk}=0 \tag{8.2}$$

is called a *Landsberg space*.
For this space, S. Numata ([13], Theorem 1) proved the beautiful theorem, that is,

Theorem 8.1. *A Landsberg space* $F_n(n>2)$ *of scalar curvature* $K\neq 0$ *is a Riemannian space of constant curvature.*

On the other hand, M. Matsumoto [10] showed that the first curvature tensor of Cartan in a four-dimensional Finsler space is written in the form

$$S_{hijk}=h_{hj}U_{ik}+h_{ik}U_{hj}-j|k\ , \tag{8.3}$$

where $U_{ik}=S_{ik}-(1/4)Sh_{ik}$, $S_{ik}:=S_i{}^m{}_{km}=S_{ki}$, $S:=S_{rs}g^{rs}$. So, we shall give the following

Definition 8.3 (cf. [12]). A Finsler space $F_n(n>4)$ satisfying the form (8.3) is called an *S4-like* Finsler space.
When a **P*-Finsler space is at the same time an *R*3-like one, substituting (8.1) into (2.4)b), we have

$$2\lambda^2 S_{hijk}=h_{hj}A_{ik}+h_{ik}A_{hj}-j|k\ , \tag{8.4}$$

where $A_{ik}=m_{ik}-b_{ik}-ch_{ik}$.

In the case $\lambda\neq 0$, which means that the space in consideration is not a Landsberg space, it follows from (8.4) that we obtain the follwoing

Theorem 8.2. *An R3-like (non-Landsberg) *P-Finsler space is S4-like.*

Next, we assume that $\lambda=0$, which means that the space in consideration is an $R3$-like Landsberg space. In this case, from (8.4), we get $A_{ik}=0$, that is,

$$m_{ik}-b_{ik}=ch_{ik}\ . \tag{8.5}$$

Substituting (8.5) and (8.2) into (2.4)a), we obtain

$$b_kC_{hij}-j|k=0\ . \tag{8.6}$$

Transvection of (8.6) with h^{hi} yields

$$b_jC_k=b_kC_j\ ,$$

where $C_k:=C^m_{km}$. Consequently, there exists a scalar function ρ such that $b_j=\rho C_j$. Substituting this relation into (8.6), we have

$$\rho C_kC_{hij}-j|k=0\ .$$

Therefore, we must consider two cases. The one is

$$C_kC_{hij}-j|k=0\ . \tag{8.7}$$

In this case, transvecting (8.7) with h^{hk}, we get

$$C^mC_{mij}=C_iC_j\ , \tag{8.8}$$

where $C^m:=g^{mi}C_i$. Also, transvection of (8.7) with C^k gives, with (8.8) in mind,

$$C^2C_{hij}=C_hC_iC_j\ ,$$

where $C^2:=C^mC_m$. The above equation implies $S_{hijk}=0$.

The other case is $\rho=0$. In this case, we have $b_j=0$, hence $b_{hj}=0$. Thus, from (8.5), we have $m_{ik}=mh_{ik}$. Consequently, taking account of Theorems 5.5 and 8.1, we can state

Theorem 8.3. *An R3-like Landsberg space is a Finsler space satisfying $S_{hijk}=0$, or a Riemannian space of constant curvature.*

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